Hierarchical exact controllability of semilinear parabolic equations

Luciano Cipriano da Silva

Instituto Federal do Rio Grande do Norte

Abstract

We present some exact controllability results for parabolic equations in the context of hierarchic control using Stackelberg-Nash strategies. We analyze two cases: in the first one, the main control (the leader) acts in the interior of the domain and the secondary controls (the followers) act on small parts of the boundary; in the second one, we consider a leader acting on the boundary while the followers are of the distributed kind. In both cases, for each leader an associated Nash equilibrium pair is found; then, we obtain a leader that leads the system exactly to a prescribed (but arbitrary) trajectory. We consider linear and semilinear problems.

Keywords: Nash equilibria, exact controllability, parabolic equations, Stackelberg-Nash strategy, Carleman inequalities.

Introduction: the problems

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with boundary Γ of class C^2. Let $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2 \subset \Omega$ be (small) nonempty open sets and let S, S_1 and S_2 be nonempty open subsets of Γ. Given $T > 0$, we will set $Q := \Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$.

In the sequel, we denote by $\nu = \nu(x)$ the outward unit normal to Ω at the point $x \in \Gamma$ and C stands for a generic positive constant. For any $m \geq 1$, the usual scalar product and norm in $L^2(\Omega)^m$ will be respectively denoted by $(\cdot\,,\cdot)$ and $\|\cdot\|$. We will consider parabolic systems of the form

$$\begin{cases} y_t - \Delta y + a(x,t)y = F(y) + f1_{\mathcal{O}} & \text{in } Q, \\ y = v^1\rho_1 + v^2\rho_2 & \text{on } \Sigma, \\ y(\cdot\,,0) = y^0 & \text{in } \Omega \end{cases} \tag{1}$$

and

$$\begin{cases} p_t - \Delta p + a(x,t)p = F(p) + u^1 1_{\mathcal{O}_1} + u^2 1_{\mathcal{O}_2} & \text{in } Q, \\ p = g\rho & \text{on } \Sigma, \\ p(\cdot\,,0) = p^0 & \text{in } \Omega, \end{cases} \tag{2}$$

where y^0, p^0, f, g, v^i and u^i are given in appropriate spaces, $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz-continuous function and $\rho, \rho_i \in C^2(\Gamma)$, with

$$0 < \rho \leq 1 \ \text{ on } \ S, \quad \rho = 0 \ \text{ on } \ \Gamma \setminus S, \quad 0 < \rho_i \leq 1 \ \text{ on } \ S_i, \quad \rho_i = 0 \ \text{ on } \ \Gamma \setminus S_i.$$

In this paper, 1_A denotes the characteristic function of the set A.

We will analyze the exact controllability to the trajectories of (1) and (2) following hierarchic control techniques, as introduced by J.-L. Lions [7]. More precisely, we will apply the Stackelberg-Nash method, which combines optimization techniques of the Stackelberg kind and non-cooperative Nash optimization techniques.

Introduction: the Stackelberg-Nash method

In order to explain the methodology, we will be initially concerned with (1).

Let $\mathcal{O}_{1,d}$ and $\mathcal{O}_{2,d}$ be nonempty open subsets of Ω and let us define the secondary cost functionals

$$J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - \xi_{i,d}|^2 \, dx\, dt + \frac{\mu_i}{2} \iint_{S_i \times (0,T)} |v^i|^2 \, d\sigma\, dt, \quad i = 1, 2, \tag{3}$$

where the $\xi_{i,d}$ are given in $L^2(\mathcal{O}_{i,d} \times (0,T))$ and α_i, μ_i are positive constants.

Let us also introduce the main functional

$$J(f) := \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |f|^2 \, dx\, dt.$$

First, for each choice of the leader f, we try to find controls v^1 and v^2, depending on f, which "minimize simultaneously" J_1 and J_2 in the following sense:

$$J_1(f; v^1, v^2) = \min_{\hat{v}^1} J_1(f; \hat{v}^1, v^2), \qquad J_2(f; v^1, v^2) = \min_{\hat{v}^2} J_2(f; v^1, \hat{v}^2). \tag{4}$$

Any pair (v^1, v^2) satisfying (4) is called a *Nash equilibrium* for J_1 and J_2 associated to f. Then, assuming that a Nash equilibrium exists for each leader, we look for a control $f \in L^2(\mathcal{O} \times (0,T))$, such that

$$J(f) = \min_{\hat{f}} J(\hat{f}), \tag{5}$$

subject to the exact controllability condition

$$y(\cdot\,,T) = \bar{y}(\cdot\,,T) \ \text{ in } \ \Omega. \tag{6}$$

Our goal is thus to prove that triplets $(f;\, v^1,\, v^2)$ of this kind exist.

In what concerns system (2), the secondary cost functionals are defined as follows:

$$K_i(g; u^1, u^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |p - \zeta_{i,d}|^2 \, dx\, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,T)} |u^i|^2 \, dx\, dt, \quad i = 1, 2, \tag{7}$$

where again the $\zeta_{i,d} = \zeta_{i,d}(x,t)$ are given functions and α_i, μ_i are positive constants.

For each leader g, we will find an associated Nash equilibrium for the cost functionals K_i, that is, a couple $(u^1,\, u^2)$ such that

$$K_1(g; u^1, u^2) = \min_{\hat{u}^1} K_1(g; \hat{u}^1, u^2), \qquad K_2(g; u^1, u^2) = \min_{\hat{u}^2} K_2(g; u^1, \hat{u}^2). \tag{8}$$

Let us set

$$K(g) := \frac{1}{2} \|g\|^2_{H^{1/2,1/4}(S\times(0,T))},$$

then, we will look for a control $g \in H^{1/2,1/4}(S \times (0,T))$ verifying

$$K(g) = \min_{\hat{g}} K(\hat{g}), \tag{9}$$

subject to

$$p(\cdot\,,T) = \bar{p}(\cdot\,,T) \quad \text{in} \quad \Omega. \tag{10}$$

Main results

Theorem 1

Suppose $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, $i = 1, 2$. Assume that one of the following conditions holds: either

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} \qquad \text{and} \qquad \xi_{1,d} = \xi_{2,d} \tag{11}$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{12}$$

If the μ_i/α_i $(i = 1, 2)$ are large enough and $F \in W^{1,\infty}(\mathbb{R})$, there exists a positive function $\varsigma = \varsigma(t)$ blowing up at $t = T$ with the following property: if $\bar{y}$ is a trajectory of (1) associated to the initial state $\bar{y}^0 \in L^2(\Omega)$ and

$$\iint_{\mathcal{O}_{i,d}\times(0,T)} \varsigma^2 |\bar{y} - \xi_{i,d}|^2 \, dx\, dt < +\infty, \quad i = 1, 2, \tag{13}$$

then, for any $y^0 \in L^2(\Omega)$ there exist controls $f \in L^2(\mathcal{O} \times (0,T))$ and associated Nash quasi-equilibria (v^1, v^2) such that the corresponding solutions to (1) satisfy $y(T,\cdot) = \bar{y}(T,\cdot)$.

Theorem 2

Suppose that

$$S \subset \overline{\mathcal{O}}_i \quad \text{and} \quad \overline{\mathcal{O}}_i \cap \overline{\mathcal{O}}_{j,d} = \emptyset, \quad i,\, j = 1, 2. \tag{14}$$

If the μ_i/α_i are large enough and $F \in W^{1,\infty}(\mathbb{R})$, there exists a positive function $\hat{\varsigma} = \hat{\varsigma}(t)$ blowing up at $t = T$ with the following property: if $\bar{p}$ is a trajectory of (2) associated to the initial state $\bar{p}^0 \in L^2(\Omega)$ and the $\zeta_{i,d} \in L^2(\mathcal{O}_{i,d} \times (0,T))$ are such that

$$\iint_{\mathcal{O}_{i,d}\times(0,T)} \hat{\varsigma}^2 |\bar{p} - \zeta_{i,d}|^2 \, dx\, dt < +\infty, \quad i = 1, 2, \tag{15}$$

then, for any $p^0 \in L^2(\Omega)$, there exist a control $g \in H^{1/2,1/4}(S \times (0,T))$ and an associated Nash quasi-equilibria (u^1, u^2) such that the corresponding solution to (2) satisfies $p(T,\cdot) = \bar{p}(T,\cdot)$.

Sketch of the proof of Theorems

We got an optimality system that characterizes the Nash equilibrium. Then we consider a associated linearized system.

From the standard controllability-observability duality theory,we know that the null controllability of optimality system together with the continuous dependence of the control with respect to the data is equivalent to an observability inequality for the solutions to the adjoint system. In this way, we deduce the observability inequality.

Finally, we apply a standard fixed-point argument.

References

References

[1] F. D. Araruna, E. Fernández-Cara, S. Guerrero, M. C. Santos, *New results on the Stackelberg-Nash exact controllability of linear parabolic equations*, Systems Control Lett., 104 (2017), 78–85.

[2] F. D. Araruna, E. Fernández-Cara, M. C. Santos, *Stackelberg-Nash exact controllabilty for linear and semilinear parabolic equations*, ESAIM: Control Optim. Calc. Var., 21 (2015), 835–856.

[3] J. I. Díaz, J.-L. Lions, *On the approximate controllability of Stackelberg-Nash strategies,* Ocean Circulation and pollution Control: A Mathematical and Numerical Investigations, (Madrid, 1997), 17–27, Springer, Berlin, 2004.

[4] A. V. Fursikov, O. Y. Imanuvilov, *Controllability of Evolution Equations*, Lectures Notes Series, Vol. 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[5] O. Y. Imanuvilov, J.-P. Puel, M. Yamamoto, *Carleman estimates for parabolic equations with nonhomogeneous boundary conditions,* Chin. Ann. Math. Ser. B, 30 (4) (2009), 333–378.

[6] O. Y. Imanuvilov, M. Yamamoto,*Carleman estimate for a parabolic equation in a sobolev space of negative order and its applications,* in: Control of Nonlinear Distributed Parameter Systems, CRC Press, 2001.

[7] J.-L. Lions, *Hierarchic Control,* Proc. Indian Acad. Sci. Math. Sci., 104 (4) (1994), 295–304.