A nonlinear perturbations of a integrodifferential equation

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Introduction

In this work, we consider the class of integrodifferential Schrödinger equations

$$-\mathcal{L}_K u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}, \tag{P}$$

where V and f are functions that satisfy mild conditions, and $-\mathcal{L}_K u$ stands for the integrodifferential operator is defined by

$$-\mathcal{L}_K u(x) = 2P.V. \int_{\mathbb{R}} (u(x) - u(y))K(x,y)\, \mathrm{d}y.$$

Here $K(x,y) = K(x-y)$ and belongs to a class of singular symmetric kernels and $P.V.$ means "in the principal value sense".
Several papers have studied problem (P) when $K(x) = C_{N,s}|x|^{-(N+2s)}$, where

$$C(N,s) = \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\zeta_1)}{|\zeta|^{N+2s}} \mathrm{d}\zeta \right)^{-1},$$

that is, when $-\mathcal{L}_K$ is the fractional Laplacian operator $(-\Delta)^s$, $0 < s < 1$.

Motivation

Many works considered nonlinearities involving polynomial growth of subcritical type in terms of the Sobolev embedding.
In the borderline case $N = 2s$, this is, $N = 1$ and $s = 1/2$, Sobolev embedding states that $H^{1/2}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for any $q \in [2, +\infty)$, but $H^{1/2}(\mathbb{R})$ is not continuous embedded in $L^\infty(\mathbb{R})$
In this case, the maximal growth which allows us to treat this problem type variationally in $H^{1/2}(\mathbb{R})$ is motivated by the Trudinger-Moser inequality

$$\sup_{\substack{u \in H^{1/2}(\mathbb{R}) \\ \|u\|_{1/2} \le 1}} \int_{\mathbb{R}} (e^{\alpha u^2} - 1)\, \mathrm{d}x < \infty \quad \alpha \in [0, \pi]. \tag{1}$$

Motivaded by (1), M. de Souza and Y. Araújo [1] studied the existence of positive solutions for fractional Schrödinger equations of the form

$$(-\Delta)^{1/2} u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}, \tag{2}$$

where V is a potential bounded and the nonlinear term $f(x,u)$ has the critical exponential growth generalizing the results in [2] by C. O. Alves, J. M. do Ó and O. H. Miyagaki, who address the problem of the standard Laplacian operator.

A Periodic Problem

Initially we study the following problem

$$\begin{cases} -\mathcal{L}_K u + V_0(x)u = f_0(x,u) & \text{in} \quad \mathbb{R}, \\ u \in X_0 \quad \text{and} \quad u \ge 0. \end{cases} \tag{P_0}$$

We assume $K : \mathbb{R} \setminus \{0\} \to \mathbb{R}^+$ is a measurable function with the properties
(K_1) $\gamma K \in L^1(\mathbb{R})$, where $\gamma(x) = min\{1, |x|^2\}$;
(K_2) there exists $\lambda > 0$ such that $K(x) \ge \lambda |x|^{-2}$, for all $x \in \mathbb{R} \setminus \{0\}$;
(K_3) $K(x) = K(-x)$, $\forall x \in \mathbb{R} \setminus \{0\}$.
Here, we assume $f_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has critical exponential growth in s, that is, there exists $\alpha_0 > 0$ such that

$$\lim_{|s| \to +\infty} f_0(x,s) e^{-\alpha s^2} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ +\infty, & \text{for all } \alpha < \alpha_0, \end{cases} \tag{3}$$

uniformly in $x \in \mathbb{R}$ and $f_0(x,s) = 0$ for all $(x,s) \in \mathbb{R} \times (-\infty, 0]$. The assumptions on the functions $V_0(x)$ and $f_0(x,u)$ are the following:
(V_0) $V_0 : \mathbb{R} \to (0, +\infty)$ is a continuous $1-$periodic function;
$(f_{0,1})$ $f_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous $1-$periodic function in x;
$(f_{0,2})$ $\lim_{s \to 0} \frac{f_0(x,s)}{s} = 0$ uniformly in $x \in \mathbb{R}$;
$(f_{0,3})$ there exists a constant $\theta > 2$ such that

$$0 < \theta F_0(x,s) := \theta \int_0^s f_0(x,t)\, \mathrm{d}t \le s f_0(x,s)$$

for all $(x,s) \in \mathbb{R} \times (0, +\infty)$;
$(f_{0,4})$ for each fixed $x \in \mathbb{R}$, the function $f_0(x,s)/s$ is increasing with respect to $s \in \mathbb{R}$;
$(f_{0,5})$ there are constants $p > 2$ and $C_p > 0$ such that

$$f_0(x,s) \ge C_p s^{p-1}, \quad \text{for all} \quad (x,s) \in \mathbb{R} \times [0, +\infty),$$

with

$$C_p > \left[\frac{(p-2)\theta\alpha_0}{(\theta - 2)p\omega} \right]^{(p-2)/2} S_p^p.$$

The Functional Setting

We define

$$X_0 := \left\{ u \in L^2(\mathbb{R});\ (u(x) - u(y))K(x,y)^{\frac{1}{2}} \in L^2(\mathbb{R}^2) \right\}$$

endowed with the norm

$$\|u\|_{X_0} = \left([u]_{1/2,K}^2 + \|u\|_{2,V_0}^2 \right)^{1/2}.$$

Lemma 1. *Assume that (V_0) and (K_2) hold, then the space X_0 is embedded in $H^{1/2}(\mathbb{R})$ and*

$$\|u\|_{1/2} \le C(\lambda, m_0) \|u\|_{X_0}, \ \forall\, u \in X_0.$$

Preliminary Results

Lemma 2. *Assume (V_0), then there exists ω such that if $0 < \alpha < \omega$, then one has a constant $C = C(\omega) > 0$, such that*

$$\sup_{\substack{u \in X_0 \\ \|u\|_{X_0} \le 1}} \int_{\mathbb{R}} (e^{\alpha u^2} - 1) \mathrm{d}x \le C(\omega). \tag{4}$$

Moreover, for any $\alpha > 0$ and $u \in X_0$, we have

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) \mathrm{d}x < \infty. \tag{5}$$

We say that $u \in X_0$ is a weak solution for (P_0) if the following equality holds:

$$\int_{\mathbb{R}^2} (u(x) - u(y))(v(x) - v(y))K(x,y)\, \mathrm{d}x\, \mathrm{d}y + \int_{\mathbb{R}} V_0(x) uv\, \mathrm{d}x = \int_{\mathbb{R}} f_0(x,u) v\, \mathrm{d}x,$$

for all $v \in X_0$.
In order to apply the mountain-pass theorem without the Palais-Smale condition to find a nontrivial solution for problem (P_0), we will consider the functional $I_0 : X_0 \to \mathbb{R}$ given by

$$I_0(u) = \frac{1}{2} \|u\|_{X_0}^2 - \int_{\mathbb{R}} F_0(x,u)\, \mathrm{d}x.$$

A Nonperiodic Problem

The second problem that we study is the following,

$$\begin{cases} -\mathcal{L}_K u + V(x)u = f(x,u) & \text{in} \quad \mathbb{R}, \\ u \in X_1 \quad \text{and} \quad u \ge 0, \end{cases} \tag{P_1}$$

(V_1) $V : \mathbb{R} \to [0, +\infty)$ is a continuous function satisfying the conditions that $V(x) \le V_0(x)$ for any $x \in \mathbb{R}$ and $V_0(x) - V(x) \to 0$ as $|x| \to \infty$;
The nonlinearity $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (3), $f(x,s) = 0$ for all $(x,s) \in \mathbb{R} \times (-\infty, 0]$ and the following conditions
(f_1) $f(x,s) \ge f_0(x,s)$ for all $(x,s) \in \mathbb{R} \times [0, +\infty)$, and for all $\varepsilon > 0$, there exists $\eta > 0$ such that for $s \ge 0$ and $|x| \ge \eta$,

$$|f(x,s) - f_0(x,s)| \le \varepsilon e^{\alpha_0 s^2};$$

(f_2) $\lim_{s \to 0} \frac{f(x,s)}{s} = 0$ uniformly in $x \in \mathbb{R}$;
(f_3) there exists a constant $\mu \ge \theta > 2$ such that

$$0 < \mu F(x,s) := \mu \int_0^s f(x,t)\, \mathrm{d}t \le s f(x,s),$$

for all $(x,s) \in \mathbb{R} \times (0, +\infty)$;
(f_4) for each fixed $x \in \mathbb{R}$, the function $f(x,s)/s$ is increasing with respect to $s \in \mathbb{R}$;
(f_5) at least one of the nonnegative continuous functions $V_0(x) - V(x)$ and $f(x,s) - f_0(x,s)$ is positive on a set of positive measure.

Main Results

Theorem 1

Assume that (V_0) and $(f_{0,1})$-$(f_{0,5})$ hold. Then (P_0) has a nonnegative and nontrivial solution.

Theorem 2

Assume that (V_1) and $(f_{0,1}) - (f_5)$ hold. Then (P_1) has a nonnegative and nontrivial solution.

References

[1] Souza M. de Araújo Y. L. *On nonlinear perturbations of a periodic fractional Schrödinger equation with critical exponential growth.* Mathematische Nachrichten. 2016; 289: 610–625.

[2] Alves C.O., do Ó J.M., Miyagaki O. *On nonlinear perturbations of a periodic elliptic problem in $\mathbb{R}^2$ involving critical growth.* Nonlinear Analysis. 2004; 56: 781–791

[3] Araújo Y., Barboza E. and de Carvalho G.; *On nonlinear perturbations of a periodic integrodifferential equation with critical exponential growth.* Applicable Analysis, 2021, 24pp.