Hierarchical control of the semi-linear heat equation with boundary controls.

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Introduction

This work is a continuation of the paper NEW RESULTS CONCERNING THE HIERARCHICAL CONTROL OF LINEAR AND SEMILINEAR PARABOLIC EQUATIONS. Bianca M.R. Calsavara, Enrique Fernández-Cara, Luz de Teresa, Jose Antonio Villa where the hierarchical control problem for boundary data is solved in the linear case.

The hierarchical control problem.

Let Ω be an open set in the n-dimensional euclidean space, with boundary Γ. Let ω ⊆ Ω an open proper subset and let leader control subset control and γ ⊂ Γ open in the relative topology named secondary control region. Denote by Q = Ω × (0, T) and Σ = Γ × (0, T). Given an initial datum y₀ ∈ L²(Ω) and a real function F define the initial value problem for the heat equation

\[ \begin{align*}
  y_t - Δy + F(y) &= v_1 y, \quad \text{in } Q \\
  y &= f_1, \quad \text{in } Σ \\
  y(0) &= y_0, \quad \text{in } Ω
\end{align*} \]

(0.0.1)

Now for suitable functions φ, φ₁, φ₂ with domain in Q consider the weighted spaces

\[ Y = \{ y : y ∈ L²(Ω) \} \quad F = \{ f : f ∈ L²(γ \times (0, T)) \} \]

(0.0.2)

We consider the following hierarchical control process:

1. Given a leader control v ∈ V find a follower control f[v] ∈ F that solves the null controllability problem, i.e., for a given positive time T the solution y₁ verifies y(T) = 0.
2. Then, look for an admissible leader control v ∈ V that minimises the functional given by

\[ P(f ; v) = \frac{1}{2} \int_Ω |y - y_0|^2 \, dx \, dt + \frac{1}{2} \int_Σ |\varphi|² \, dx \, dt \]

(0.0.3)

where Q := Ω × (0, T), the set Ω ⊂ Ω is an open set on Rⁿ and the function y₀ ∈ L²(Q₀).

Basic tools: Carleman inequalities.

Then, let us introduce the weight functions

\[ \sigma(x, t) = \frac{e^{\lambda t} |y| - e^{\lambda t} f_1(x, t) \varphi_0}{|\varphi|}, \quad \xi(x, t) = \frac{e^{\lambda t} |y| - e^{\lambda t} f_1(x, t) \varphi_1}{|\varphi|} \]

(0.0.4)

where t ∈ C¹([0, T)] satisfies t(½) ≥ T/4 in [0, T/2] and (t(0) + T/2) in [T/2, T] and λ > 0 are large constants. This constant λ and s will be fixed in a convenient way. Let us introduce the weights g = e^{sλ} f₁(0) = (sλ)⁻¹ f₁(0) φ₁ = (sλ)⁻¹ f₁(0) φ₂ = (sλ)⁻¹ f₁(0) φ₂. With these definitions state the next theorem.

Theorem 0.1. Given a function p ∈ P and a bilinear form B : P₀ × P₀ → ℝ defined by

\[ B(p, q) = \int_Ω g_0^2 \partial_ψ^2 p \partial_ψ^2 q \, dx + \int_Σ \xi \partial_ψ^2 p \partial_ψ^2 q \, ds \]

(0.0.5)

The functional B defines a norm on P₀ and the closure is denoted by P. There exist positive constants k₀ and k₁, only depending on Ω, γ, and T, such that, if we take λ = k₀ and s ≥ k₁, any p ∈ P satisfies

\[ \int_Ω |(\xi + 1)^{1/2} |p|² + |\varphi(2)|² + |\nabla \varphi|^2 |p|² \leq C₁ B(p,p,p) \]

Furthermore, λ₁ and s₁ can be found arbitrarily large.

The linear case

Proposition 0.1. A fixed positive time T, consider a potential a ∈ L²(Ω). For a leader control v ∈ V and y₀ ∈ L²(Ω) it exists a follower control f[v] ∈ F such that y(T) = 0 when y is a solution to

\[ \begin{align*}
  y_t - Δy + F(y) &= v_1 y, \quad \text{in } Q \\
  y &= f_1, \quad \text{in } Σ \\
  y(0) &= y_0, \quad \text{in } Ω
\end{align*} \]

(0.0.6)

Moreover, it exists a function p ∈ P such that the solution and the control to

\[ \begin{align*}
  f[v] &= -g_0^2 \partial_ψ^2 p_1, \quad y = -g_0^2 L₁(p)
\end{align*} \]

(0.0.7)

where p solves the integral equation

\[ \int_Ω g_0^2 L₁(p) \varphi \, dx \, dt + \int_Σ \xi \partial_ψ^2 p \partial_ψ^2 q \, ds = \int_Σ vq \, ds + \int_Σ v_0 \, ds \]

(0.0.8)

for any function q ∈ P.

Proof. Null controllability \( \iff \int_{Ω} f[v] = \frac{1}{2} \int_Ω |y|^² \, dx \, dt = \frac{1}{2} \int_Σ \xi \, \varphi \, ds \).

The semi-linear case

Theorem 0.1. Let a leader control v ∈ V and a positive time T > 0. Then there exist a follower control f[v] ∈ F that steers y(T) = 0. Where y ∈ Ω solves the initial value problem

\[ \begin{align*}
  y_t - Δy + F(y) &= v_1 y, \quad \text{in } Q \\
  y &= f_1, \quad \text{in } Σ \\
  y(0) &= y_0, \quad \text{in } Ω
\end{align*} \]

(0.0.9)

Moreover is possible to get the explicit form

\[ f[v] = -g_0^2 \partial_ψ^2 p_1, \quad y = g_0^2 L₁(p) \]

(0.0.10)

where p is a solution to

\[ \int_Ω g_0^2 L₁(p) \varphi \, dx \, dt + \int_Σ \xi \partial_ψ^2 p \partial_ψ^2 q \, ds = \int_Σ vq \, ds + \int_Σ v_0 \, ds \]

(0.0.11)

for any q ∈ P. Moreover it is possible to get the estimation

\[ \|f[v]|x + |y| ≤ C \|1 + \|y\|_{L²(Ω)} \] .

(0.0.12)

Proof. Define z ∈ L²(Ω)

\[ \begin{align*}
  y &= y_0 - Δy + F(y) = v_1 y, \quad \text{in } Q \\
  y &= f_1, \quad \text{in } Σ \\
  y(0) &= y_0, \quad \text{in } Ω
\end{align*} \]

(0.0.13)

The main result

Theorem 0.2. For all a pair (f[v], v) ∈ P there exist a following control f[v] in F that fulfills the null controllability problem (the state y(T) = 0) and the leader v minimises the functional P. Moreover the pair (f[v], v) ∈ P given by

\[ \begin{align*}
  y &= y_0 - Δy + F(y) = v_1 y, \quad \text{in } Q \\
  y &= f_1, \quad \text{on } Σ \\
  y(0) &= y_0, \quad \text{in } Ω
\end{align*} \]

(0.0.14)

where p ∈ P solves the equation

\[ \int_Ω g_0^2 L₁(p) \varphi \, dx \, dt + \int_Σ \xi \partial_ψ^2 p \partial_ψ^2 q \, ds = \int_Σ vq \, ds + \int_Σ v_0 \, ds \]

(0.0.15)

Define v to the solution to

\[ \begin{align*}
  y &= y_0 - Δy + F(y) = v_1 y, \quad \text{in } Q \\
  y &= f_1, \quad \text{on } Σ \\
  y(0) &= y_0, \quad \text{in } Ω
\end{align*} \]

(0.0.16)

with \( \psi \) ∈ P the unique solution to

\[ \int_Ω g_0^2 \psi \, dx \, dt + \int_Σ \xi \partial_ψ^2 p \partial_ψ^2 q \, ds = -\int_Σ \xi \partial_ψ^2 p \partial_ψ^2 dΣ \quad \forall q ∈ P \]

(0.0.17)

Then, the leader control is characterized by

\[ v = -g_0^2 \gamma \partial_ψ \phi \]

(0.0.18)

Both controls in the boundary

The function

\[ \begin{align*}
  y &= y_0 - Δy + F(y) = 0 \quad \text{in } Q \\
  y &= f_1, \quad \text{in } Σ \\
  y(0) &= y_0, \quad \text{in } Ω
\end{align*} \]

(0.0.19)

The leader control has regularity in L¹(0, T, H¹/²(Ω)).

References