Hierarchical control of the semi-linear heat equation with boundary controls.

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Introduction

This work is a continuation of the paper NEW RESULTS CONCERNING THE HIERARCHICAL CONTROL OF LINEAR AND SEMILINEAR PARABOLIC EQUATIONS,Bianca M.R. Calsavara, Enrique Fernandez-Cara, Luz de Teresa, Jose Antonio Villa where the hierarchical control problem for boundary data is solved in the linear case.

The hierarchical control problem.

Let Ω be an open set in the n-dimensional euclidean space, with boundary Γ. Let $\omega \subset \Omega$ an open proper subset called **leader control subset** and $\gamma \subset \Gamma$ open in the relative topology named **secondary control region.** Denote by $Q = \Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$. Given an initial datum y_0 in $L^2(\Omega)$ and a real function F define the initial value problem for the heat equation

$$\begin{cases} y_t - \Delta y + F(y) = v1_\omega & \text{in } Q \\ y = f1_\gamma & \text{in } \Sigma \\ y(0) = y_0 & \text{in } \Omega \end{cases} \tag{0.0.1}$$

Now for suitable functions $\varrho, \varrho_0, \varrho_1$ with domain in Q consider the weighted spaces

$$\mathcal{Y} = \{y : \varrho y \in L^2(Q)\} \quad \mathcal{F} = \{f : \varrho_0 f \in L^2(\gamma \times (0,T))\}$$
$$\mathcal{V} = \{v : \varrho_0 v \in L^2(\omega \times (0,T))\} \tag{0.0.2}$$

We consider the following *hierarchical control process*:

1. Given a leader control v in $\mathcal{V}$ find a follower control $f[v]$ in $\mathcal{F}$ that solves the null controllability problem, i.e for a given positive time T the solution y to (0.0.1) verifies $y(T) = 0$.
2. Then, we look for an admissible leader control $v \in \mathcal{V}$ that minimises the functional given by

$$P(f;v) = \frac{\alpha}{2}\int_{Q_d} |y - y_d|^2\, dxdt + \frac{1}{2}\int_{\omega\times(0,T)} \varrho_0^2 |v|^2\, dxdt \tag{0.0.3}$$

where $Q_d := \Omega_d \times (0,T)$, the set $\Omega_d \subset \Omega$ is an open set on $\mathbb{R}^n$ and the function $y_d \in L^2(Q_d)$.

Basic tools: Carleman inequalities.

Then, let us introduce the weight functions

$$\tilde{\sigma}(x,t) := \frac{e^{4\lambda\|\tilde{\eta}^0\|_\infty} - e^{\lambda(2\|\tilde{\eta}^0\|_\infty + \tilde{\eta}^0(x))}}{\ell(t)}, \quad \tilde{\xi}(x,t) := \frac{e^{\lambda(2\|\tilde{\eta}^0\|_\infty + \tilde{\eta}^0(x))}}{\ell(t)}, \tag{0.0.4}$$

where $\ell \in C^2([0,T])$ satisfies $\ell(t) \geq T^2/4$ in $[0,T/2]$ and $\ell(t) = t(T-t)$ in $[T/2,T]$ and $\lambda, s > 0$ are large enough. This constants λ and s will be fixed in a convenient way. Let us introduce the weights $\varrho = e^{s\tilde{\sigma}}$, $\varrho_0 = (s\tilde{\xi})^{-3/2}\lambda^{-2}\varrho$, $\varrho_1 = (s\tilde{\xi})^{-1/2}\lambda^{-1}\varrho$, $\varrho_2 = (s\tilde{\xi})^{1/2}\varrho$. With this definitions state the next theorem.

Let $\mathcal{P}_0 = \{q \in C^2(Q) : q|_\Sigma = 0\}$ and give a bilinear form $B : \mathcal{P}_0 \times \mathcal{P}_0 \longrightarrow \mathbb{R}$ defined by

$$B(a,p,q) = \int_Q \varrho^{-2} L_a^*(p) L_a^*(q)\, dxdt + \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta p \partial_\eta q\, d\sigma dt \tag{0.0.5}$$

The functional B defines a norm on $\mathcal{P}_0$ and the closure is denoted by $\mathcal{P}$.

There exist positive constants λ_0, s_0 and C_1, only depending on Ω, γ and T, such that, if we take $\lambda = \lambda_0$ and $s \geq s_0$, any $p \in \mathcal{P}$ satisfies

$$\iint_Q \left[\varrho_2^{-2}(|p_t|^2 + |\Delta p|^2) + \varrho_1^{-2}|\nabla p|^2 + \varrho_0^{-2}|p|^2\right] \leq C_1\, B(0;p,p).$$

Furthermore, λ_1 and s_1 can be found arbitrarily large.

The linear case

Theorem (Theorems as blocks)

Proposition 0.1. *Fixed a positive time T, consider a potential $a \in L^\infty(Q)$. For a leader control $v \in \mathcal{V}$ and $y_0 \in L^2(\Omega)$ it exists a follower control $f[v] \in \mathcal{F}$ such that $y(T) = 0$ where y is a solution to*

$$\begin{cases} y_t - \Delta y + ay = v1_\omega & \text{in } Q \\ y = f[v]1_\gamma & \text{in } \Sigma \\ y(0) = y_0 & \text{in } \Omega \end{cases} \tag{0.0.6}$$

Moreover, it exists a function $p \in \mathcal{P}$ such that the follower control and the solution to (0.0.6) are characterised in the form

$$f[v] = -\varrho_0^{-2}\partial_\eta p 1_\gamma; \quad y = \varrho^{-2} L_a^*(p) \tag{0.0.7}$$

where p solves the integral equation

$$\int_Q \varrho^{-2} L_a^*(p) L_a^*(q)\, dxdt + \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta p\, \partial_\eta q\, d\Sigma = \int_{\omega\times(0,T)} vq + \langle y_0, q(0)\rangle_{L^2(\Omega)} \tag{0.0.8}$$

for any function $q \in \mathcal{P}$.

Proof.

$$\text{Null controllability} \iff \inf_{f\in\mathcal{F}}\left(S(f;v) = \frac{1}{2}\int_Q \varrho^2 |y|^2\, dxdt + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2 |f|^2\, d\Sigma \right) \tag{0.0.9}$$

□

The semi-linear case

Theorem 0.1. *Let a leader control $v \in \mathcal{V}$. and a positive time $T > 0$. Then there exist a follower control $f[v] \in \mathcal{F}$ that steers $y(T) = 0$. Where $y \in \mathcal{Y}$ solves the initial value problem*

$$\begin{aligned} &y_t - \Delta y + F(y) = v1_\omega && \text{in } Q \\ &y = f[v]1_\gamma && \text{in } \Sigma \\ &y(0) = y_0 && \text{in } \Omega \end{aligned} \tag{0.0.10}$$

Moreover is possible to get the explicit form

$$f[v] = -\varrho_0^{-2}\partial_\eta p|_\gamma; \quad y = \varrho^{-2} L_{F'(y)}^*(p) \tag{0.0.11}$$

where p is a solution to

$$\int_Q \varrho^{-2} L_{F'(y)}^*(p) L_{F_0(y)}^*(q) + \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta p \partial \eta q\, d\Sigma = \int_{\omega\times(0,T)} vq + \int_\Omega y_0 q(0)\, dx. \tag{0.0.12}$$

for any $q \in \mathcal{P}$. Moreover it is possible to get the estimation

$$\|f[v]\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \leq C\left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right). \tag{0.0.13}$$

Proof. Define $z \in L^2(Q)$

$$\begin{cases} y_t - \Delta y + F_0(z)y = v1_\omega & \text{in } Q \\ y = f1_\gamma & \text{in } \Sigma \\ y(0) = y_0 & \text{in } \Omega \end{cases} \tag{0.0.14}$$

. where $y_z \in H^{1/2,1/4}(Q)$. □

Main result

Theorem 0.2. *It exists a pair $(\hat{f}[\hat{v}], \hat{v}) \subset \mathcal{G}$ such that the follower control $\hat{f}[\hat{v}]$ fulfils the null controllability problem (the state $\hat{y}(T) = 0$) and the leader $\hat{v}$ minimises the functional P. Moreover the pair $(\hat{f}[\hat{v}], \hat{v})$ is given by*

$$\begin{cases} \hat{y}_t - \Delta\hat{y} + F(\hat{y}) = \hat{v}1_\omega \text{ in } Q, \\ \hat{y} = \hat{f}[\hat{v}]1_\gamma \text{ on } \Sigma, \\ \hat{y}(\cdot,0) = y_0 \text{ in } \Omega, \end{cases} \tag{0.0.15}$$

$$\hat{f}[\hat{v}] = -\varrho_0^{-2}\partial_\eta\hat{p}\big|_\gamma, \quad \hat{y} = \varrho^{-2} L_{F'(\hat{y})}^*\hat{p}, \tag{0.0.16}$$

where $\hat{p} \in \mathcal{P}$ solves the equation

$$\int_Q \varrho^{-2} L_{F'(\hat{y})}^*\hat{p}\, L_{F_0(\hat{y})}^* p' + \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta\hat{p}\partial_\eta p'\, d\Sigma = \int_{\omega\times(0,T)} \hat{v}p' + \int_\Omega y_0(x)p'(x,0)\, dx \quad \forall p' \in \mathcal{P}. \tag{0.0.17}$$

Define $\hat{\gamma}$ the solution to

$$\begin{cases} -\hat{\gamma}_t - \Delta\hat{\gamma} + F'(\hat{y})\hat{\gamma} = \alpha(\hat{y} - y_d)1_{\Omega_d} + F'(\hat{y})\hat{\phi} + \varrho^{-2}F''(\hat{y})\hat{p}L_0^*\hat{\phi} \text{ in } Q, \\ \hat{\gamma} = 0 \text{ on } \Sigma, \\ \hat{\gamma}(\cdot,T) = 0 \text{ in } \Omega, \end{cases} \tag{0.0.18}$$

with $\hat{\phi} \in \mathcal{P}$ the unique solution to

$$\int_Q \varrho^{-2} L_0^*(\hat{\phi}) L_{F'(y)}^*(q)\, dxdt + \int_\Sigma \varrho_0^{-2}\partial_\eta\hat{\phi}\partial_\eta q\, d\Sigma = -\int_\Sigma \varrho_0^{-2}\partial_\eta\hat{\gamma}\partial_\eta q\, d\Sigma \quad \forall q \in \mathcal{P}. \tag{0.0.19}$$

Then, the leader control is characterized by

$$v = -\varrho_0^{-2}(\hat{\gamma} + \hat{\phi})\big|_{\omega\times(0,T)}. \tag{0.0.20}$$

Both controls in the boundary

Define the equation

$$\begin{cases} y_t - \Delta y + F(y) = 0 & \text{in } Q \\ y = f1_\gamma + v\chi_\sigma & \text{in } \Sigma \\ y(0) = y_0 & \text{in } \Omega \end{cases} \tag{0.0.21}$$

The leader control has regularity in $L^2(0,T;H^{1/2}(\Omega))$.

References

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