Nondegenerate solutions for constrained problems

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Abstract

Let $\overline{M}^n$ be a connected compact smooth manifold (with or without boundary), where $n \geq 2$. We prove that nondegeneracy (in the sense of Morse theory) of nonconstant solutions for a class of singularly perturbed semilinear elliptic problems on $\overline{M}$ is generic with respect to the pair $(\epsilon, g) \in]0, \infty[\times \mathcal{M}^k$, where $\mathcal{M}^k$ is the set of Riemannian metrics in $\overline{M}$ of class C^k $(k \geq 1)$. As applications, we show that under certain growth conditions, such result generalizes to nondegeneracy of any solution for the Allen-Cahn or nonlinear Schrödinger equations.

Context and main result

Let $f, c \colon \mathbb{R} \to \mathbb{R}$ be functions of class C^2, where $c \not\equiv 0$. Given $\epsilon > 0$ and $g \in \mathcal{M}^k$, we are interested in weak solutions $(u, \lambda) \in H_g(M) \times \mathbb{R}$ to the following constrained semilinear problem with homogeneous Neumann boundary condition:

$$\begin{cases} -\epsilon^2 \Delta_g u = f(u) + \lambda c'(u) & \text{in } M, \\ \partial_\nu u = 0 & \text{on } \partial M := \overline{M} \setminus M, \\ C_g(u) := \int_M c(u) \, d\mu_g = 1, \end{cases} \tag{$P_{\epsilon,g}$}$$

where Δ_g is the Laplace-Beltrami operator $\Delta_g := \operatorname{tr} \nabla \circ \operatorname{grad}_g$, μ_g is the Radon measure on $\overline{M}$ induced by $g \in \mathcal{M}^k$ and $H_g(M)$ is the Hilbert space obtained as completion of $C^\infty(M)$ with respect to the inner product $\langle u, v \rangle_g := \int_M g(\nabla u, \nabla v) + uv \, d\mu_g$.

A weak solution $(u, \lambda) \in H_g(M) \times \mathbb{R}$ to $(P_{\epsilon,g})$ is said to be *nondegenerate* when the only solution to the following linearized problem is the trivial one:

$$\begin{cases} -\epsilon^2 \Delta_g v = [f'(u) + \lambda c''(u)]v + \Lambda c'(u) & \text{on } M, \\ \int_M c'(u) v \, d\mu_g = 0, \\ (v, \Lambda) \in H_g(M) \times \mathbb{R}. \end{cases} \tag{$Q_{\epsilon,g,u,\lambda}$}$$

Problem $(P_{\epsilon,g})$ is variational for a certain functional $J_{\epsilon,g} \colon H_g(M) \times \mathbb{R} \to \mathbb{R}$ of class C^1. Considering such functional, our definition of nondegeneracy coincides with the Morse theoretic notion of a nondegenerate critical point of $J_{\epsilon,g}|_{C_g^{-1}(1) \times \mathbb{R}}$.

Suppose that f, c and their derivatives have subcritical growth, i.e., there are $C > 0$ and $p \in [2, p_n[$ for which given $t \in \mathbb{R}$,

$$|f(t)|, \ |c'(t)| \leq C(1 + |t|^{p-1}); \tag{1}$$

$$|f'(t)|, \ |c''(t)| \leq C(1 + |t|^{p-2}); \tag{2}$$

where $p_n := \infty$ for $n = 2$, $p_n := (2n)/(n-2)$ for $n \geq 3$.

By identifying the set of constant functions with $\mathbb{R}$, we can state our main result as

Theorem ([1, Theorem A])

Fix $g_0 \in \mathcal{M}^k$. If $f, c \in C^2(\mathbb{R})$ satisfy the growth conditions (1) and (2), then

$$\mathcal{D} = \big\{ (\epsilon, g) \in]0, \infty[\times \mathcal{M}^k : \text{ if } (u, \lambda) \in (H_{g_0}(M) \setminus \mathbb{R}) \times \mathbb{R} \text{ is a weak solution to } (P_{\epsilon,g}), \text{ then } (u, \lambda) \text{ is nondegenerate} \big\}$$

is an open dense subset of $]0, \infty[\times \mathcal{M}^k$.

Application to the Allen-Cahn and nonlinear Schrödinger equations

If we can identify the constant solutions, then we can refine our main result to encompass every solution. In particular, let us consider the Allen-Cahn equation under a volume constraint:

Proposition ([1, Proposition B])

Fix $g_0 \in \mathcal{M}^k$. Given $(\epsilon, g) \in]0, \infty[\times \mathcal{M}^k$, consider the Allen-Cahn equation under a volume constraint:

$$\begin{cases} -\epsilon^2 \Delta_g u + W'(u) = \lambda, \\ \int_M u \, d\mu_g = \eta, \\ (u, \lambda) \in H_g(M) \times \mathbb{R}, \end{cases} \tag{$AC_{\epsilon,g}$}$$

where $W \in C^2(\mathbb{R})$. If $f := -W'$ satisfies the growth conditions (1) and (2), then

$$\mathcal{D}^* = \big\{ (\epsilon, g) \in]0, \infty[\times \mathcal{M}^k : \text{ if } (u, \lambda) \in H_{g_0}(M) \times \mathbb{R} \text{ is a weak solution to } (AC_{\epsilon,g}), \text{ then } (u, \lambda) \text{ is nondegenerate} \big\}$$

is an open dense subset of $]0, \infty[\times \mathcal{M}^k$.

If we consider the nonlinear Schrödinger equation under its usual constraint:

Proposition ([1, Proposition C])

Fix $g_0 \in \mathcal{M}^k$. Given $(\epsilon, g) \in]0, \infty[\times \mathcal{M}^k$, consider the Nonlinear Schrödinger equation

$$\begin{cases} -\epsilon^2 \Delta_g u + V(u) = \lambda u, \\ \int_M u^2 \, d\mu_g = 1, \\ (u, \lambda) \in H_{g_0}(M) \times \mathbb{R}, \end{cases} \tag{$NLS_{\epsilon,g}$}$$

where $V \in C^1(\mathbb{R})$. If $f := -V$ satisfies the growth conditions (1) and (2), then

$$\mathcal{D}^* = \big\{ (\epsilon, g) \in]0, \infty[\times \mathcal{M}^k : \text{ if } (u, \lambda) \in H_{g_0}(M) \times \mathbb{R} \text{ is a weak solution to } (NLS_{\epsilon,g}), \text{ then } (u, \lambda) \text{ is nondegenerate} \big\}$$

is an open dense subset of $]0, \infty[\times \mathcal{M}^k$.

Sketching the proof of [1, Theorem A]

The proof of [1, Theorem A] is inspired by the general argument present in [3], where Micheletti and Pistoia are interested in the genericity of nondegeneracy for solutions to

$$\begin{cases} -\epsilon^2 \Delta_g u + u = u|u|^{p-2} & \text{in } M, \\ u \in H_g(M) \end{cases} \tag{3}$$

with respect to the parameter $(\epsilon, g) \in]0, 1[\times \mathcal{M}^k$.

Let $X = Z = H_{g_0}(M) \times \mathbb{R}$, $Y = V =]0, \infty[\times \mathcal{S}^k$, $U = (H_{g_0}(M) \setminus \mathbb{R}) \times \mathbb{R}$ and $z_0 = (0, -1)$, where $\mathcal{S}^k$ is the Banach space of symmetric 2-covectors on $\overline{M}$ of class C^k. The key step to prove [1, Theorem A] is the following classical transversality theorem:

Theorem ([2, Theorem 5.4])

Let X, Y, Z be real Banach spaces and U, V be respective open subsets of X, Y. Let $F \colon V \times U \to Z$ be a map of class C^m, where $m \geq 1$. Let $z_0 \in \operatorname{im} F$. Suppose that

1. Given $y \in V$, $F(y, \cdot) \colon x \mapsto F(x, y)$ is a Fredholm map of index $l < m$, i.e., $dF(y, \cdot)_x \colon X \to Z$ is a Fredholm operator of index l for any $x \in U$;
2. z_0 is a regular value of F, i.e., $dF_{(y_0, x_0)} \colon Y \times X \to Z$ is surjective for any $(y_0, x_0) \in F^{-1}(z_0)$;
3. Let $\iota \colon F^{-1}(z_0) \to Y \times X$ be the canonical embedding and $\pi_Y \colon Y \times X \to Y$ be the projection of the first coordinate. Then $\pi_Y \iota \colon F^{-1}(z_0) \to Y$ is σ-proper, i.e., $F^{-1}(z_0) = \bigcup_{s=1}^{\infty} C_s$, where given $s = 1, 2, \ldots$, C_s is a closed subset of $F^{-1}(z_0)$ and $\pi_Y \iota|_{C_s}$ is proper.

In this context, the set $\{y \in V : z_0 \text{ is a regular value of } F(y, \cdot)\}$ is an open dense subset of V.

In fact, we take $F \colon]0, \infty[\times \mathcal{M}^k \times (H_{g_0}(M) \setminus \mathbb{R}) \times \mathbb{R} \to H_{g_0}(M) \times \mathbb{R}$ as given by

$$F(\epsilon, g, u, \lambda) = \left(u - A_{\epsilon,g} B(u, \lambda), -\int_M c(u) \, d\mu_g \right),$$

where $B \colon H_{g_0}(M) \times \mathbb{R} \to L_{g_0}^{p'}(M)$ is the Nemytskii operator given by

$$B(u, \lambda) = u + f(u) + \lambda c'(u)$$

and $A_{\epsilon,g}$ is the adjoint of the canonical inclusion $H_{\epsilon,g}(M) \hookrightarrow L_g^p(M)$ and $H_{\epsilon,g}(M)$ is the Hilbert space $H_g(M)$ endowed with the inner product $\langle u, v \rangle_{\epsilon,g} := \int_M \epsilon^2 g(\nabla u, \nabla v) + uv \, d\mu_g$.

Constant functions are an obstruction to item 2 in the previous theorem, hence their exclusion. In [3], this phenomenom is translated in the exclusion of the constant solution 1 in [3, Theorem 1.1]. The cause of this phenomenom is a strong continuation theorem which asserts that solutions which are constant in a nonempty open subset of $\overline{M}$ are constant in the whole manifold $\overline{M}$.

References

References

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