A sharp integral inequality for closed spacelike submanifolds immersed in the de Sitter space

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Motivation

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- As for the case of de Sitter space, Goddard (1977) [11] conjectured that every complete spacelike hypersurface with constant mean curvature H in de Sitter space $\mathbb{S}_1^{n+1}$ should be totally umbilical.
- In 1987, Ramanathan [16] prove that Goddard's conjecture is true for $\mathbb{S}_1^3$ and $0 \leqslant H \leqslant 1$. However, for $H > 1$ he showed that the conjecture is false, as it can be seen from an example due to Dajczer and Nomizu (1981) in [10].

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- Simultaneously and independently, Akutagawa (1987) [2] also proved that Goddard's conjecture is true when either $n = 2$ and $H^2 \leqslant 1$ or $n \geqslant 3$ and $H^2 < \frac{4(n-1)}{n^2}$.

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- In [14], Montiel (1988) proved that Goddard's conjecture is true provided that M^n is closed (that is, compact and without boundary).

- In [15], Montiel (1996) characterized the hyperbolic cylinders as the only complete noncompact spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature $H^2 = 4(n-1)/n^2$ and having at least two ends.

Motivation

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- Meanwhile, Alías and Romero (1995) [5] introduced a new method to study n-dimensional closed spacelike submanifolds in de Sitter space $\mathbb{S}_q^{n+p}$ of index q $(1 \leqslant q \leqslant p)$ by means of certain integral formulas which have a very clear geometric meaning.

Motivation

- More recently, the first and second authors jointly with Alías (Mediterr. J. Math. (2018)) [3] also investigated complete spacelike submanifolds M^n immersed in $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector field and constant scalar curvature R.

Motivation

- More recently, the first and second authors jointly with Alías (Mediterr. J. Math. (2018)) [3] also investigated complete spacelike submanifolds M^n immersed in $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector field and constant scalar curvature R.

- Next, Alías and Meléndez (Mediterr. J. Math. (2020)) [4] studied the rigidity of closed hypersurfaces with constant scalar curvature isometrically immersed in the unit Euclidean sphere $\mathbb{S}^{n+1}$. In particular, they established a sharp integral inequality for the behavior of the norm of the traceless second fundamental form, with the equality characterizing the totally umbilical hypersurfaces and the Clifford tori $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$.

Motivation

In our paper, we extend the techniques of [3] and [4] in order to establish a sharp integral inequality for a closed spacelike submanifold M^n with constant scalar curvature immersed with parallel normalized mean curvature vector field in the de Sitter space $\mathbb{S}_p^{n+p}$, and we use it to characterize totally umbilical round spheres $\mathbb{S}^n(r)$ of $\mathbb{S}_1^{n+1} \hookrightarrow \mathbb{S}_p^{n+p}$.

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We define the mean curvature vector field h and the mean curvature function H of M^n, respectively by

$$h = \frac{1}{n}\sum_\alpha \left(\sum_i h_{ii}^\alpha\right) e_\alpha \quad \text{and} \quad H = |h| = \sqrt{\sum_\alpha \left(\sum_i h_{ii}^\alpha\right)^2}.$$

Using the structure equations, we obtain the Gauss equation

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha). \tag{1}$$

From (1), we conclude that the Ricci curvature and the (normalized) scalar curvature of M^n are given, respectively, by

$$R_{ij} = (n-1)\delta_{ij} - \sum_{\alpha}\left(\sum_{k} h^{\alpha}_{kk}\right) h^{\alpha}_{ij} + \sum_{\alpha,k} h^{\alpha}_{ik} h^{\alpha}_{kj} \tag{2}$$

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From (2) and (3) we obtain

$$|A|^2 = n^2 H^2 + n(n-1)(R-1), \tag{4}$$

where $|A|^2 = \sum_{\alpha,i,j}(h_{ij}^{\alpha})^2$ is the square of the length of the second fundamental form A of M^n.

Throughout this work, we will consider the case that $H > 0$. So, we can choose a local orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ such that $e_{n+1} = \frac{h}{H}$. Thus,

$$H^{n+1} = \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^{\alpha} = \frac{1}{n}\mathrm{tr}(h^{\alpha}) = 0,\ \alpha \geqslant n+2, \tag{5}$$

where $h^{\alpha} = (h^{\alpha}_{ij})$ denotes the second fundamental form of M^n in direction e_{α} for every $n+1 \leqslant \alpha \leqslant n+p$.

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where $h^{\alpha} = (h_{ij}^{\alpha})$ denotes the second fundamental form of M^n in direction e_α for every $n+1 \leqslant \alpha \leqslant n+p$.

We define on M^n the symmetric tensor $\Psi = \sum_{i,j=1}^{n} \psi_{ij}\omega_i \otimes \omega_j$, where $\psi_{ij} = nH\delta_{ij} - h_{ij}^{n+1}$. According to Cheng and Yau [9], we consider an operator L associated to Ψ acting on any smooth function $f \in \mathcal{C}^2(M)$ in the following way

The Cheng-Yau's operator

$$Lf = \sum_{i,j=1}^{n} \psi_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}^{n+1}) f_{ij} = nH\Delta f - \sum_{i,j} h_{ij}^{n+1} f_{ij}, \qquad (6)$$

where f_{ij} stands for a component of the Hessian of f.

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where f_{ij} stands for a component of the Hessian of f. Thus,

$$Lf = \mathrm{tr}(P \circ \nabla^2 f), \qquad (7)$$

where

$$P = nHI - h^{n+1},$$

I is the identity in the algebra of smooth vector fields on M^n, $h^{n+1} = (h_{ij}^{n+1})$ denotes the second fundamental form of M^n in direction e_{n+1} and $\nabla^2 f$ stands for the self-adjoint linear operator metrically equivalent to the Hessian of f.

Ellipticity of L

Lemma 2.1.

Let M^n be a spacelike submanifold in the de Sitter space $\mathbb{S}_p^{n+p}$ with $H > 0$. Let μ_- and μ_+ be, respectively, the minimum and the maximum of the eigenvalues of the operator P at every point $p \in M^n$. If $R < 1$ (resp., $R \leqslant 1$) on M^n, then P is positive definite (positive semi-definite) and the operator L is elliptic (resp., semi-elliptic), with

$$\mu_- > 0 \quad (\textit{resp.}, \mu_- \geqslant 0).$$

and

$$\mu_+ < 2nH \quad (\textit{resp.}, \mu_+ \leqslant 2nH).$$

Total umbilicity tensor

We will also deal with the following symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \Phi_{ij}^{\alpha} \omega_i \otimes \omega_j e_{\alpha}, \tag{8}$$

where $\Phi_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij}$, and H^{α} is defined in (5).

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Let $|\Phi|^2 = \sum_{\alpha,i,j}(\Phi_{ij}^{\alpha})^2$ be the square of the length of Φ. It is easy to check that Φ is traceless and, from (4), we get the following relation

$$|\Phi|^2 = |A|^2 - nH^2 = n(n-1)H^2 + n(n-1)(R-1). \tag{9}$$

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Moreover $|\Phi|^2 \geqslant 0$, with equality at the umbilical points of M^n. For that reason Φ is usually called the total umbilicity tensor of M^n.

Lemma 2.2.

Let M^n be a spacelike submanifold in $\mathbb{S}_p^{n+p}$, with parallel normalized mean curvature vector field and constant scalar curvature $R \leqslant 1$. Then

$$\frac{1}{2}L(|\Phi|^2) \geqslant \frac{1}{\sqrt{n(n-1)}}|\Phi|^2 Q_{R,n,p}(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(1-R)},$$

where the real function $Q_{R,n,p}$ is

$$Q_{R,n,p}(x) = \frac{(n-p-1)}{p}x^2 - (n-2)x\sqrt{x^2 + n(n-1)(1-R)} + n(n-1)R.$$

Main result

Theorem 3.1.

Let M^n be a closed spacelike submanifold immersed in $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector field and constant normalized scalar curvature $R \leqslant 1$. Then

$$\int_M |\Phi|^{q+2} Q_{R,n,p}(|\Phi|) dM \leqslant 0, \quad \forall q > 2; \tag{10}$$

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Moreover, assuming that $0 < R < 1$, the equality holds in (10) if, and only if, M^n is a totally umbilical round sphere $\mathbb{S}^n(r)$, with $r = \frac{1}{R} > 1$ immersed in $\mathbb{S}_1^{n+1} \hookrightarrow \mathbb{S}_p^{n+p}$.

Proof

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From Lemma 2.2 we have that

$$L(|\Phi|^2) \geqslant \frac{2}{\sqrt{n(n-1)}}|\Phi|^2 Q_{R,n,p}(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(1-R)}. \qquad (11)$$

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Now, let us take $u = |\Phi|^2$. So, (11) can be rewritten as

$$L(u) \geqslant \frac{2}{\sqrt{n(n-1)}} u Q_{R,n,p}(\sqrt{u})\sqrt{u + n(n-1)(1-R)}. \qquad (12)$$

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Taking into account that $u \geqslant 0$, $R \leqslant 1$ and observing that when $R = 1$ (9) guarantees that $u > 0$, from (12) we get

$$u^{\frac{q+2}{2}} Q_{R,n,p}(\sqrt{u}) \leqslant \frac{\sqrt{n(n-1)}}{2} \frac{u^{\frac{q}{2}}}{\sqrt{u + n(n-1)(1-R)}} L(u), \qquad (13)$$

for every real number q.

Proof

By the compactness of M^n, we can integrate both sides of (13) in order to obtain

$$\int_M u^{\frac{q+2}{2}} Q_{R,n,p}(\sqrt{u})dM \leqslant \frac{\sqrt{n(n-1)}}{2} \int_M \frac{u^{\frac{q}{2}}}{\sqrt{u+n(n-1)(1-R)}} L(u)dM. \tag{14}$$

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But, from (7) we have

$$f(u)L(u) = \mathrm{div}(f(u)P(\nabla u)) - f'(u)\langle P(\nabla u), \nabla u\rangle, \tag{15}$$

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for every smooth function $f \in \mathcal{C}^1(\mathbb{R})$. Integrating both sides of (15) and using Stokes' theorem, we deduce that

$$\int_M f(u)L(u)dM = -\int_M f'(u)\langle P(\nabla u), \nabla u\rangle dM, \tag{16}$$

for every smooth function f.

Proof

Thus,

$$\int_M u^{\frac{q+2}{2}} Q_{R,n,p}(\sqrt{u})dM \leqslant -\frac{\sqrt{n(n-1)}}{2}\int_M f'(u)\langle P(\nabla u), \nabla u\rangle dM. \quad (17)$$

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In our case, for every real number $q > 2$, we choose

$$f(t) = \frac{t^{q/2}}{\sqrt{t+n(n-1)(1-R)}}, \quad \text{for } t \geqslant 0. \quad (18)$$

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Hence, assuming $R \leqslant 1$ and that $R = 1$ only for $t > 0$, we get

$$f'(t) = \frac{(q-1)t^{q/2} + n(n-1)(1-R)qt^{\frac{q-2}{2}}}{2\left(t + n(n-1)(1-R)\right)^{3/2}} \geqslant 0, \quad (19)$$

for every real number $q > 2$.

Proof

Using (18) and (19) into (17), we can estimate

$$\int_M u^{\frac{q+2}{2}} Q_{R,n,p}(\sqrt{u})dM \leqslant -\frac{\sqrt{n(n-1)}}{2}\int_M f'(u)\langle P(\nabla u), \nabla u\rangle dM \leqslant 0, \quad (20)$$

since we know that the operator P is positive semi-definite.

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since we know that the operator P is positive semi-definite.

Therefore, we conclude

$$\int_M |\Phi|^{q+2} Q_{R,n,p}(|\Phi|)dM \leqslant 0.$$

This proves the inequality (10).

Proof

Furthermore, if the equality holds in (10), from (20) we get

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$$f'(u) = \frac{(q-1)u^{q/2} + n(n-1)(1-R)qu^{\frac{q-2}{2}}}{2\left(u + n(n-1)(1-R)\right)^{3/2}} \geqslant 0 \tag{22}$$

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with equality if and only if $q > 2$ and $u = 0$. Consequently, if

$$\langle P(\nabla u), \nabla u\rangle = 0.$$

since P is positive definite taking into account Lemma 2.1, we get that $\nabla u = 0$ on M^n.

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In the case that $|\Phi| = 0$, we can reason as in the last part of the proof of Theorem 1.3 of Guo, X., Li (2013) [12] to conclude that M^n must be a totally umbilical round sphere $\mathbb{S}^n(r)$, with $r = \frac{1}{R} > 1$, immersed in a totally geodesic de Sitter space $\mathbb{S}_1^{n+1} \hookrightarrow \mathbb{S}_p^{n+p}$.

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Indeed, let N_1 be the sub-bundle spanned by $\{e_{n+2}, \cdots, e_{n+p}\}$. Then, from our assumption $\nabla^{\perp} e_{n+1} = 0$ it follows that N_1 is parallel in the normal bundle. Besides, we get that $|\Phi^{\alpha}|^2 = \sum_{i,j} (\Phi_{ij}^{\alpha})^2 = 0$ for each $n+2 \leqslant \alpha \leqslant n+p$, which means that M^n is totally geodesic with respect to N_1. Hence, from Theorem 1 of Yau, S.T. (1974) [17] we obtain the desired conclusion.

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As in the last part of the proof of Theorem 1.2 in [4], we have that $|\Phi| = u_0$ is such that $Q_{R,n,p}(u_0) = 0$ because of

$$\int_M |\Phi|^{q+2} Q_{R,n,p}(|\Phi|) dM = 0.$$

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Consequently, we can apply Theorem 1 of [3] obtaining that $p = 1$, $n \geqslant 3$ and that M^n should be isometric to a hyperbolic cylinder $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$ of radius $r > 0$.

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Therefore, since we are assuming that M^n is closed, we conclude that this second case cannot occur. □

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