The geometry of self- similar solutions of a geometric flow

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and let $\psi : \Sigma^n \to \mathbb{R}_1^{n+1}$ be an isometric spacelike immersion in the Minkowski space.

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$$\begin{cases} \dfrac{\partial \Psi}{\partial t} = (H - \mathcal{H})\eta, \\ \Psi(0, p) = \psi(p). \end{cases} \tag{1}$$

§1. Introduction

Many authors have studied this geometric flow on Minkowski space and Lorentzian manifolds aiming to understand the short and long behavior of its solutions and the relationship with the problem of prescribed mean curvature or how to foliating spacetimes by almost null like hypersurfaces., e.g., Ecker-Huisken in 1991, Ecker in 1993 and 1997, Huisken-Yau in 1996, Aarons 2005, and many others important works.

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An interesting case in the mean curvature flow equation is the non-parametric setting, where each slice is a graph, that is,

$$\Sigma_t^n = \{(U(t,\cdot), p) : p \in \Omega\} \subset \mathbb{R}_1 \times \mathbb{R}^n = \mathbb{R}_1^{n+1},$$

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for $U(t,\cdot) \in C^1(\Omega)$ and with $|DU(t,\cdot)| < 1$ in Ω. With a straightforward computation we can verify that in this setting the equation (1) is equivalent to the equation:

$$\frac{\partial U}{\partial t} = \sqrt{1 - |DU(t,\cdot)|^2}\left[\operatorname{div}\left(\frac{DU}{\sqrt{1 - |DU(t,\cdot)|^2}}\right) - \mathcal{H}\right], \qquad (2)$$

which is the parabolic analogue of the maximal surface equation.

Aarons proved that the solution of the equation (2) exists for all t and assuming that $\mathcal{H} = c > 0$, $\Sigma_0 = \Psi(0, \Sigma)$ has bounded curvature and it never intersects future null infinity $\mathcal{I}^+$ or past null infinity $\mathcal{I}^-$ at all, then $\Sigma_t = \Psi(t, \Sigma)$ converges under the flow to a convex graph satisfying

$$H = -a\nu + c, \tag{3}$$

for $a < 0$ and $\nu = \langle N, e_{n+1} \rangle$

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As the solutions of the equation (2) model the behavior of the MCF with forcing term $\mathcal{H} = c$ at infinity, such equation became very interesting. Such solutions are called *downward translating soliton* and are the functions $u \in C^\infty(\mathbb{R}^n)$ satisfying the quasilinear elliptic PDE

$$\mathrm{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = c + \frac{a}{\sqrt{1-|Du|^2}}. \tag{4}$$

§1. Introduction

In this talk we use new concepts introduced in the literature by some authors, for example, Alías, de Lira, Martín, Rigoli and others, and on this new set up we establish new necessary conditions to the existence of translating soliton and we also comment about the existence of solutions of such equation.

§2. Set Up

Throughout the talk, let us consider an $(n+1)$-dimensional Lorentzian product space $\overline{M}^{n+1}$ of the form $\mathbb{R}_1 \times \mathbb{P}^n$, where $(\mathbb{P}^n, g_{\mathbb{P}^n})$ is an n-dimensional connected Riemannian manifold and $\overline{M}^{n+1}$ is endowed with the standard product metric $\langle\, , \rangle = -\pi^*_{\mathbb{R}}(dt^2) + \pi^*_{\mathbb{P}^n}(g_{\mathbb{P}^n})$, where $\pi_{\mathbb{R}}$ and $\pi_{\mathbb{P}^n}$ denote the canonical projections from $\mathbb{R}_1 \times \mathbb{P}^n$ onto each factor.

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Lets consider

- $\psi : \Sigma^n \to \overline{M}^{n+1}$ a spacelike immersion in $\overline{M}^{n+1}$;
- $\eta \in \mathfrak{X}(\Sigma)^{\perp}$ a timelike unit vector field having the same time-orientation of ∂_t;
- $A(X) = -\overline{\nabla}_X \eta$ the second fundamental form associated to N;
- $H = trace(A)$ the mean curvature function on Σ;
- $\nu = \langle \eta, \partial_t \rangle$ the angle function (or lapse function);

§2. Set Up

Following *de Lira and Martín*, we introduce the notation:

Definition

A spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in a Lorentzian product space $\overline{M}^{n+1} = \mathbb{R}_1 \times \mathbb{P}^n$ is a *spacelike translating soliton of the mean curvature flow with respect to* ∂_t with forcing term $\mathcal{H} = c$ and *soliton constant* $a \in \mathbb{R}_*$ if its mean curvature function satisfies

$$H = -a\nu + c. \tag{5}$$

In case $a < 0$ we call the hypersurface by *spacelike downward translating soliton*.

§2. Set Up

A graph $\Sigma(u)$ is a spacelike translating soliton with respect to ∂_t with forcing term $\mathcal{H} = c$ and soliton constant a if, and only if, $u \in C^\infty(\Omega)$ is a solution of the following system:

$$\begin{cases} \mathrm{div}_{\mathbb{P}^n}\left(\dfrac{Du}{W}\right) = \dfrac{a}{W} + c \\ \\ |Du|_{\mathbb{P}^n} < 1, \end{cases} \tag{6}$$

and here $W = \sqrt{1 - |Du|^2_{\mathbb{P}^n}}$. If $u \in C^\infty(\mathbb{P})$ is a solution of the system (6), $\Sigma(u)$ is called an *entire spacelike translating graph*.

We introduce a notation in $\mathbb{R}_1 \times \mathbb{P}^n$ in a similar way as in $\mathbb{R}_1^{n+1}$, assuming that $\mathbb{P}$ is complete:

Definition

Let $u : \mathbb{P}^n \to \mathbb{R}$ be a solution of (6). The spacelike hypersurface defined by *u intersect the null infinity* $\mathcal{I}$ if $\lim_{p \to \infty} |Du|_{\mathbb{P}^n}(p) = 1$.

Note that such notation is interesting only in the non-compact case.

§3. Mean Convexity

Here we recall a classical result due to Omori in 60's. The result is the following:

Theorem (Omori)

Let Σ be a connected and complete Riemannian manifold whose sectional curvature has a lower bound. If u is a smooth function such that $u^* = \sup_\Sigma u < \infty$, then there is a sequence $\{p_n\}$ of points on Σ satisfying:

$$(i)\ u(p_n) \to u^*;\ \ (ii)\ |\nabla u|(p_n) \to 0;\ and\ (iii)\ (\mathrm{Hess}\ u)(p_n)(X, X) < \frac{1}{n}|X|^2.$$

§3. Mean Convexity

Under suitable conditions we prove:

Theorem

Let $\mathbb{R}_1 \times \mathbb{P}^n$ be a Lorentz space whose its base is complete, has sectional curvature bounded from below and nonnegative Ricci curvature. Thus, any entire spacelike downward translating soliton which never intersect the null infinity is mean convex.

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A simple and interesting consequence is the following result:

Corollary

Under the conditions of the previous theorem. There is no entire spacelike downward translating soliton which never intersect the null infinity and forcing term greater than minus the soliton constant.

Example

Consider the surface

$$\Sigma = \{(-a \ln y, x, y) : y > 0\} \subset \mathbb{R}_1 \times \mathbb{H}^2,$$

where the constant $a \in \mathbb{R}^*$ is such that $0 < |a| < 1$ and $\mathbb{H}^2 = \mathbb{R} \times \mathbb{R}_+$ stands for the two-dimensional hyperbolic space endowed with the complete metric $\langle\, , \,\rangle_{\mathbb{H}^2} = \frac{1}{y^2}(dx^2 + dy^2)$. This is a complete spacelike translating soliton with constant mean curvature $H = -\frac{a}{\sqrt{1-a^2}} = -a\,\nu$, with respect to its natural orientation. Hence, the hypothesis of nonnegative Ricci curvature on the Riemannian base $\mathbb{P}^n$ is, indeed, necessary to guarantee the nonexistence result of spacelike translating solitons.

Proof: Let $\{e_1, e_2, \cdots, e_n\}$ be a local geodesic frame at a point p of $\mathbb{P}$. Denote by $a^{ij} = \delta_{ij} + \frac{u_i u_j}{W^2}$ and $W = (1 - |Du|^2)^{\frac{1}{2}}$, and so the equation (5) can be written as $a^{ij} u_{ij} = cW + a$, here and from now on we will use the Einstein summation convention. Taking the derivative in direction e_k we get

$$a_k^{ij} u_{ij} + a^{ij} u_{ijk} = cW_k. \tag{7}$$

Observe that

$$\begin{aligned}
a_k^{ij} u_{ij} &= \left(\frac{u_{ik} u_j}{W^2} + \frac{u_i u_{jk}}{W^2} - 2\frac{u_i u_j W_k}{W^3} \right) u_{ij} \\
&= \left(\frac{u_{ik} u_j}{W^2} + \frac{u_i u_{jk}}{W^2} + 2\frac{u_i u_j u_l u_{lk}}{W^4} \right) u_{ij} \\
&= \frac{2}{W} \left(\frac{u_{ik} u_j}{W} + \frac{u_i u_j u_l u_{lk}}{W^3} \right) u_{ij} \\
&= \frac{2}{W} (-W_i u_{ik} - \frac{u_i u_j u_{jk}}{W^2} W_i) \\
&= -\frac{2}{W} \left(\delta_{ij} + \frac{u_i u_j}{W^2} \right) W_i u_{jk} \\
&= -\frac{2}{W} a^{ij} W_i u_{jk},
\end{aligned} \tag{8}$$

We also need the following computation:

$$\begin{aligned} W_{ij} &= -\frac{u_{kj}u_{ki}}{W} - \frac{u_k u_{kij}}{W} + \frac{u_k u_{ki} W_j}{W^2} \\ &= -\frac{u_k u_{kij}}{W} - \left(\frac{u_{kj}u_{ki}}{W} + \frac{u_k u_{ki} u_l u_{lj}}{W^3}\right) \\ &= -\frac{u_k u_{kij}}{W} - \frac{1}{W}\left(\delta_{kl} u_{ki} u_{lj} + \frac{u_k u_{ki} u_l u_{lj}}{W^2}\right) \\ &= -\frac{u_k u_{kij}}{W} - \frac{1}{W} a^{kl} u_{ki} u_{lj}, \end{aligned} \tag{9}$$

and recall the metric on a graph in $\mathbb{R}_1 \times \mathbb{P}$ is given by $g_{ij} = \delta_{ij} - u_i u_j$ and so its inverse is $g^{ij} = \delta_{ij} + W^{-2} u_i u_j$. Since the second fundamental form of the graph is $a_{ij} = -\frac{u_{ij}}{W}$, we deduce that $|A|^2 = \frac{1}{W^2} g^{ij} g^{kl} u_{ki} u_{lj}$.

$$
\begin{aligned}
0 &= \frac{u_k}{W}\left(-\frac{2}{W}a^{ij}W_i u_{jk} + a^{ij}u_{ijk} - cW_k\right) \\
&= -\frac{2}{W^2}a^{ij}W_i u_k u_{jk} + a^{ij}\frac{u_k u_{ijk}}{W} - c\frac{u_k W_k}{W} \\
&= -\frac{2}{W}a^{ij}W_i u_{jk}\frac{u_k}{W} + a^{ij}(u_{kij} + R_{jkis}u_s)\frac{u_k}{W} - cW_k\frac{u_k}{W} \\
&= -\frac{2}{W}a^{ij}W_i u_{jk}\frac{u_k}{W} + a^{ij}u_{kij}\frac{u_k}{W} + a^{ij}R_{jkis}u_s\frac{u_k}{W} - cW_k\frac{u_k}{W} \\
&= -\frac{2}{W}a^{ij}W_i u_{jk}\frac{u_k}{W} + a^{ij}(-W_{ij} - \frac{1}{W}a^{kl}u_{ki}u_{lj}) + a^{ij}R_{jkis}u_s\frac{u_k}{W} - cW_k\frac{u_k}{W} \\
&= -a^{ij}W_{ij} + \frac{2}{W}a^{ij}W_iW_j - \frac{1}{W}a^{ij}a^{kl}u_{ki}u_{lj} - a^{ij}R_{jksi}u_s\frac{u_k}{W} - cW_k\frac{u_k}{W} \\
&= W^2a^{ij}\left(\frac{1}{W}\right)_{ij} + cu_k\left(\frac{1}{W}\right)_k W - W|A|^2 - \frac{1}{W}\mathrm{Ric}_{\mathbb{P}}(Du, Du),
\end{aligned}
$$

we used Ricci identity in third equality, (9) in the fifth equality and the expression for the length of the second fundamental form in the last one.

Thus,

$$\frac{|A|^2}{W} = \mathcal{L}\left(\frac{1}{W}\right) + c\frac{1}{W}\left\langle Du, D\left(\frac{1}{W}\right)\right\rangle - \frac{1}{W^3}\mathrm{Ric}_{\mathbb{P}}(Du, Du),$$

for $\mathcal{L}v = a^{ij}v_{ij}$, which is an elliptic operator on $\mathbb{P}$. Since Ricci curvature of the base is nonnegative, we get

$$\frac{|A|^2}{W} \leq \mathcal{L}\left(\frac{1}{W}\right) + c\frac{1}{W}\left\langle Du, D\left(\frac{1}{W}\right)\right\rangle,$$

and using that a^{ij} is bounded from above in the quadratic form sense and the lower boundedness of the sectional curvature we are able to apply Omori's theorem for the function $\dfrac{1}{W}$ and deduce that there is a sequence $\{p_n\}$ such that $\lim_n |A|^2(p_n) = 0$, and in particular we obtain that

$$\sup_{\mathbb{P}}(-\nu) = \sup_{\mathbb{P}}\left(\frac{1}{W}\right) = -\frac{c}{a},$$

and the result follows. **Q.E.D.**

Theorem (Simons type formula)

Let Σ be a spacelike hypersurface of $\mathbb{R}_1 \times \mathbb{M}^n(\kappa)$. Then,

$$\begin{aligned} \frac{1}{2}\Delta|A|^2 &= |\nabla A|^2 + trace(A \circ \mathrm{Hess}\, H) - \kappa H\langle AT, T\rangle \\ &\quad + n\kappa|AT|^2 + n\kappa\nu^2\left(|A|^2 - \frac{H^2}{n}\right) + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2, \end{aligned} \tag{10}$$

where k_i are the principal curvatures.

§4. Second Fundamental Form of Entire Translatings

Theorem

Let Σ be an entire spacelike translating soliton in $\mathbb{R}_1 \times \mathbb{M}^n(\kappa)$ which never intersect the null infinity. Then, the length of the second fundamental form is bounded from above.

Proof: Using Theorem 7 and the hessian of ν we have

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 &= |\nabla A|^2 + (a\kappa\nu\langle AT, T\rangle - a\kappa\nu H|T|^2 + a\langle\nabla_T A, A\rangle - a\nu TrA^3) \\ &+(\kappa n|A|^2 + \kappa n|AT|^2 + \kappa|T|^2|A|^2 - HTrA^3 - \kappa H^2 - 2\kappa H\langle AT, T\rangle \\ &+|A|^4) - \kappa H\langle AT, T\rangle - \kappa\nu^2 H^2 + n\kappa|AT|^2 + n\kappa\nu^2|A|^2. \\ &= |\nabla A|^2 + \frac{a}{2}T(|A|^2) + \kappa(a\nu - 3H)\langle AT, T\rangle - (a\nu + H)TrA^3 \\ &-\kappa(1+\nu^2)H^2 + \kappa(n + |T|^2 + n\nu^2)|A|^2 + 2\kappa n|AT|^2 \\ &-a\kappa\nu H|T|^2 + |A|^4\end{aligned}$$

After many estimates we obtain

$$\frac{1}{2}\Delta|A|^2 \geq \frac{1}{2}|A|^4 - C. \qquad (11)$$

§4. Second Fundamental Form of Entire Translatings

Set $f = |A|^2$ and consider $\zeta(x) = r_0^2 - r^2$ and $g = \zeta^2 f$. So, applying (11) in $B_\Sigma(p, r_0)$, we have

$$\frac{g^2}{\zeta^4} \leq C_1 + \Delta(\zeta^{-2} g) = C_1 + \zeta^{-2}\Delta g - 2\zeta^{-3}\langle \nabla\zeta, \nabla g\rangle + g\Delta(\zeta^{-2}).$$

For $\overline{x}$ a point of maximum of g in such ball we have $\nabla g(\overline{x}) = 0$ e $\Delta g(\overline{x}) \leq 0$. Using the tracing of the Gauss equation and the boundedness of ν we obtain that $\mathrm{Ric}(v, v) \geq -\frac{\alpha^2}{4}$, where α is a constant, and using Bochner formula we get $\Delta r^2 \leq C_3(1 + r^2)$.

Thus, at $\overline{x}$, we get

$$\begin{aligned} \frac{1}{2}g^2 &\leq C_1\zeta^4 + g\zeta^4\Delta(\zeta^{-2}) = C_1\zeta^4 - 2g\zeta\Delta\zeta + 6g|\nabla\zeta|^2 \\ &\leq C_2(r_0^8 + 2g(r_0^2\Delta r^2 + 12r^2)) \\ &\leq C_4(r_0^8 + r_0^4 g), \end{aligned}$$

for r_0 sufficient large. Therefore, $g(\overline{x}) \leq C_5 r_0^4$ and making $r_0 \to \infty$ we conclude that $|A|^2 \leq C_5$, and so we finish the proof. **Q.E.D.**

Thank you for your attention!

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