Submanifolds Immersed in a Warped Product With Density

Wallace Ferreira Gomes

Unidade Acadêmica de Matemática – UAMat

Universidade Federal de Campina Grande – UFCG

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This conference corresponds to the paper

J.G. Araújo, H.F. de Lima, W.F. Gomes and M.A.L. Velásquez, *Submanifolds immersed in a warped product with density*, Bulletin of the Belgian Mathematical Society - Simon Stevin, v. 27, issue 5, 2020.

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Our approach is based on the technique developed in

Jogli G. Araújo, Henrique F. de Lima and Marco A.L. Velásquez, *Submanifolds immersed in a warped product: rigidity and nonexistence*, Proceedings of the American Mathematical Society, v. 147, p. 811-821, 2019.

Summary

1. Preliminaries
   - The Riemannian warped product $I \times_f M^{n+p}$
   - Submanifolds into $I \times_f M^{n+p}$
   - The Riemannian warped product $I \times_f M_\varphi^{n+p}$
   - The height function
2. The main result and its consequences
3. Further results
4. Liouville-type result
5. References

For a given $(n+p)$-dimensional Riemannian manifold $(M^{n+p}, \langle , \rangle_M)$ and an open interval $I \subset \mathbb{R}$, our ambient space

$$I \times_f M^{n+p}$$

is the $(n+p+1)$-dimensional product manifold $I \times M^{n+p}$ endowed with the Riemmanian warped metric

$$\langle , \rangle \;=\; dt^2 + f(t)^2 \langle , \rangle_M, \tag{1}$$

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For every $\tau \in I$, the slice

$$\mathbb{M}_\tau^{n+p} \ = \ \{\tau\} \times M^{n+p} \subset I \times_f M^{n+p}$$

is a hypersurface.

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We have that the shape operator of $\mathbb{M}_\tau^{n+p}$ is given by

$$A_\tau(v) \;=\; -\overline{\nabla}_v \partial_t \;=\; -\frac{f'(\tau)}{f(\tau)}\, v,$$

for every tangent vector v in $(\tau, x) \in \mathbb{M}_\tau^{n+p}$.

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Therefore, the correspondence

$$I \ni \tau \longmapsto \mathbb{M}^{n+p}_\tau$$

determines a foliation of $I \times_f M^{n+p}$ by totally umbilical hypersurface with constant mean curvature given by

$$\mathcal{H}(\tau) = \frac{1}{n+p}\,\mathrm{tr}(A_\tau) = -\frac{f'(\tau)}{f(\tau)}. \tag{2}$$

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$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(X, Y), \tag{3}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma^n)$.

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for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma^n)$. Here

$$\alpha : \mathfrak{X}(\Sigma^n) \times \mathfrak{X}(\Sigma^n) \to \mathfrak{X}^{\perp}(\Sigma^n)$$

stands for the vector valued second fundamental form of Σ^n, defined by

$$\alpha(X, Y) = (\overline{\nabla}_X Y)^{\perp}, \tag{4}$$

where $(\overline{\nabla}_X Y)^{\perp}$ denotes the normal component of $\overline{\nabla}_X Y$ along Σ^n.

Moreover, the Weingarten formula of Σ^n is given by

$$\overline{\nabla}_X \eta = -A_\eta(X) + \nabla^\perp_X \eta, \tag{5}$$

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The mean curvature vector field $\vec{H}$ of Σ^n is defined by

$$\vec{H} \;=\; \frac{1}{n}\mathrm{tr}(\alpha) \;=\; \frac{1}{n}\sum_{i=1}^{n} \alpha(E_i, E_i),$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on Σ^n.

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where X is a tangent vector field on Σ^n. From this, we define the drift Laplacian by

$$\Delta_\varphi u = \mathrm{div}_\varphi(\nabla u) = \Delta u - \langle \nabla u, \nabla \varphi \rangle, \tag{7}$$

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According to Gromov [5], the *weighted mean curvature vector field*, or simply *φ-mean curvature vector field*, $\vec{H}_\varphi$ of Σ^n is defined by

$$\vec{H}_\varphi = \vec{H} + \frac{1}{n}(\overline{\nabla}\varphi)^\perp, \tag{8}$$

where $\vec{H}$ denotes the standard mean curvature vector field of Σ^n defined in trace of second fundamental form and $(\overline{\nabla}\varphi)^\perp \in \mathfrak{X}^\perp(\Sigma)$ stands for the normal component of $\overline{\nabla}\varphi$ along Σ^n.

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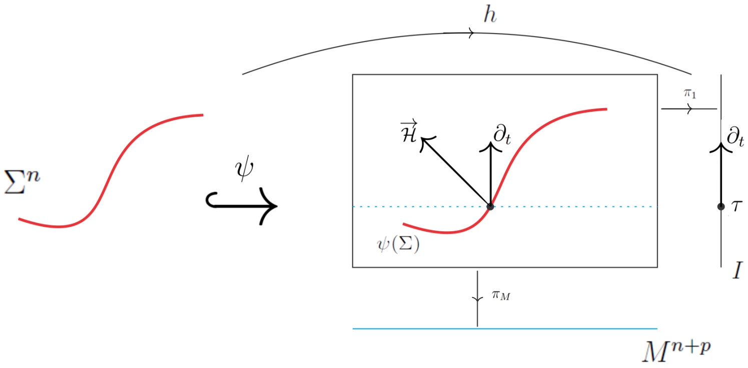
where $\partial_t = \partial_t^\top + \partial_t^\perp$. Here $\partial_t^\top \in \mathfrak{X}(\Sigma^n)$ and $\partial_t^\perp \in \mathfrak{X}^\perp(\Sigma^n)$ denote, respectively, the tangential and normal components of ∂_t.



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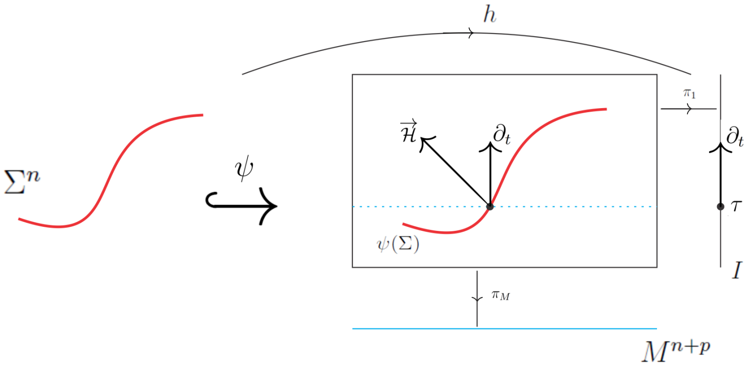
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$$\nabla u = f(h)\nabla h = f(h)\partial_t^\top = K^\top, \tag{9}$$

where $K^\top$ denotes the tangential component of the closed conformal vector field

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and tracing this expression we get

$$\Delta u \;=\; n\left(f'(h) + \langle \vec{H}, K\rangle\right) \;=\; n\left(f'(h) + f(h)\langle \vec{H}, \partial_t\rangle\right). \tag{11}$$

Since we are considering Σ^n immersed in $I \times_f M_\varphi^{n+p}$, from (7), (9) and (11) we get

$$\begin{aligned}\Delta_\varphi u &= \Delta u - \langle \nabla u, \overline{\nabla}\varphi \rangle \\ &= n(f'(h) + f(h)\langle \vec{H}, \partial_t \rangle) + f(h)\langle \partial_t^\perp, (\overline{\nabla}\varphi)^\perp \rangle.\end{aligned}$$

Thus, from (8) and (12) we obtain

$$\begin{aligned}\Delta_\varphi u &= n(f'(h) + f(h)\langle \vec{H} + \frac{1}{n}(\overline{\nabla}\varphi)^\perp, \partial_t \rangle) \qquad (12) \\ &= n(f'(h) + f(h)\langle \vec{H}_\varphi, \partial_t \rangle).\end{aligned}$$

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Lemma 1

Let Σ^n be a closed submanifold immersed in $I \times_f M_\varphi^{n+p}$. Then

(i) $\min_\Sigma \langle \vec{H}_\varphi, \partial_t \rangle \leq \mathcal{H}_\varphi(h^*)$, *where* $h^* = \max_\Sigma h$, *and*

(ii) $\max_\Sigma \langle \vec{H}_\varphi, \partial_t \rangle \geq \mathcal{H}_\varphi(h_*)$, *where* $h_* = \min_\Sigma h$.

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Let us consider on Σ^n the function $u = g(h)$. Since Σ^n is closed, the function u attains its minimum and maximum at some points $p_{\min}$ and $p_{\max}$.

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Thus,

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$$|\vec{H}_\varphi|^2 = \frac{H_{\phi,\varphi}^2 + f'(\tau)^2}{f(\tau)^2}. \tag{13}$$

Proof.

From Lemma 1 and using the fact that $(\log f)'' \geq 0$ we have

$$\min_{\Sigma}\langle \vec{H}_{\varphi}, \partial_t\rangle \leq \mathcal{H}_{\varphi}(h^*) \leq \mathcal{H}_{\varphi}(h_*) \leq \max_{\Sigma}\langle \vec{H}_{\varphi}, \partial_t\rangle. \tag{14}$$

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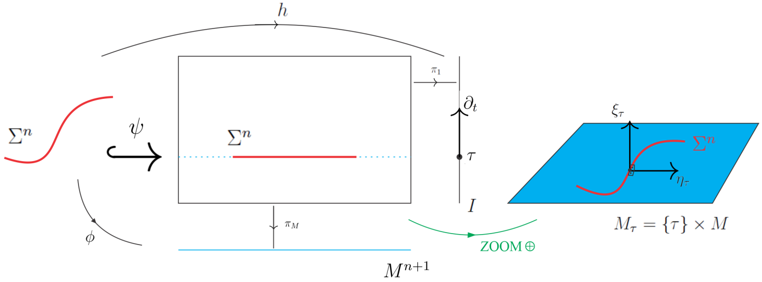
Consequently, it $p = 1$ is not difficult to see we can consider the (locally defined) unit normal vector field N of the hypersurface $\phi : \Sigma^n \to M^{n+1}$, with $\langle N, N \rangle_M = 1$.



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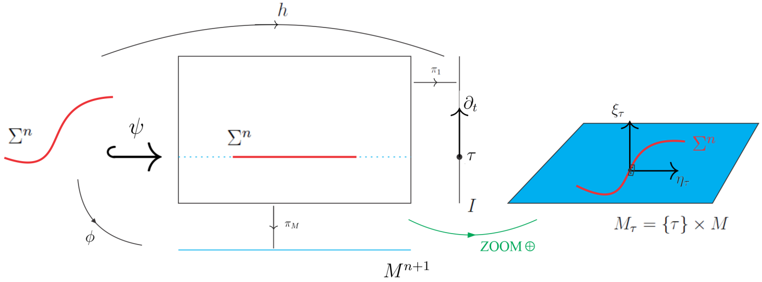

Thus, from (8) jointly with equation (4.18) of [3] and using again the assumption that φ does not depend on the parameter $t \in I$, it is not difficult to verify that holds the following equation

$$\vec{H}_{\varphi} = \frac{H_{\phi,\varphi}}{f(\tau)^2} N + \frac{f'(\tau)}{f(\tau)} \partial_t. \tag{16}$$

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From relation (13) in Theorem 1 we obtain the following nonexistence result:

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Corollary 1

The only n-dimensional closed φ-minimal submanifolds immersed in a weighted product space $\mathbb{R}^p \times M_\varphi^{n+1}$ are the closed φ-minimal hypersurfaces immersed in M_φ^{n+1}.

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From relation (13) in Theorem 1 we also obtain the following nonexistence result:

Corollary 2

There do not exist closed φ-minimal submanifolds Σ^n immersed in a weighted warped product $I \times_f M_\varphi^{n+1}$ such that $(\log f)'' \geq 0$ and f' does not vanish on I.

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Lemma 2

Let Σ^n_φ *be a complete weighted manifold whose Bakry-Émery-Ricci curvature tensor is bounded from below and let* $u : \Sigma^n \to \mathbb{R}$ *be a smooth function satisfying* $\sup_\Sigma u < +\infty$*. Then, there exists a sequence of points* $\{p_k\}_{k\in\mathbb{N}} \subset \Sigma^n$ *such that*

$$\lim_k u(p_k) = \sup_\Sigma u \quad \text{and} \quad \limsup_k \Delta_\varphi u(p_k) \le 0.$$

The previous lemma jointly with Lemma 2 enable us to obtain an extension of Lemma 1.

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Lemma 3

Let Σ^n be a complete submanifold immersed in $I \times_f M_\varphi^{n+p}$, such that its Bakry-Émery-Ricci tensor is bounded from below.

(i) *If Σ^n lies above a slice of $I \times_f M_\varphi^{n+p}$, then $\sup_\Sigma \langle \vec{H}_\varphi, \partial_t \rangle \geq \mathcal{H}_\varphi(h_*)$, where $h_* = \inf_\Sigma h \in I$;*

(ii) *If Σ^n lies below a slice of $I \times_f M_\varphi^{n+p}$, then $\inf_\Sigma \langle \vec{H}_\varphi, \partial_t \rangle \leq \mathcal{H}_\varphi(h^*)$, where $h^* = \sup_\Sigma h \in I$.*

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Proof.

The result follows from Lemma 2 and from ideas established in the proof of Lemma 1.

In our next result, we will assume that the ambient space obeys a convergence condition which was established by Montiel [1]. Before, we recall that a *slab* of a weighted warped product $I \times_f M^{n+p}_\varphi$ is just a region between two slices M_{τ_1} and M_{τ_2}, for some $\tau_1 < \tau_2$.

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Theorem 2

Let $I \times_f M^{n+p}_\varphi$ be a weighted warped product such that $(\log f)'' \geq 0$, with the equality $(\log f)'' = 0$ holding only at isolated points of I, and which obeys the following convergence condition

$$K_M \geq \sup_I (f'^2 - f f''), \tag{17}$$

where K_M stands for the sectional curvature of M^{n+p}. Suppose in addition that the Hessian of the weight function φ is bounded from below. Let $\psi : \Sigma^n \to I \times_f M^{n+p}_\varphi$ be a complete submanifold which lies in a slab of $I \times_f M^{n+p}_\varphi$, with bounded second fundamental form and such that the support function $\langle \vec{H}_\varphi, \partial_t \rangle$ is constant. Then, $\psi(\Sigma)$ is contained in a slice $\{\tau\} \times M^{n+p}$, for some $\tau \in I$. Moreover, when $p = 1$, $\phi := \pi_M \circ \psi : \Sigma^n \to M^{n+1}$ is a hypersurface with φ-mean curvature $H_{\phi,\varphi}$ satisfying (13).

Proof.

We can reason as in the proof of Theorem 1 (but using now Lemma 3 instead of Lemma 1) in order to show that

$$\mathcal{H}(h^*) = \mathcal{H}(h_*) = \langle \vec{H}_\varphi, \partial_t \rangle = \text{constant}. \tag{18}$$

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Hence, since we are assuming that Σ^n lies in a slab of $I \times_f M_\varphi^{n+p}$, $|\alpha|$ is bounded and $\operatorname{Hess}\varphi$ is bounded from below, from

$$\operatorname{Ric}_\varphi = \operatorname{Ric} + \operatorname{Hess}\varphi. \tag{19}$$

and

$$\operatorname{Ric}(\mathrm{X},\mathrm{X}) \geq -\left(n\frac{|f''(h)|}{f(h)} + |\alpha|^2\right)|X|^2. \tag{20}$$

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we get that the Bakry-Émery-Ricci tensor of Σ^n is bounded from below. Our constraint on $\log f$ implies that the function $\mathcal{H}_\varphi(t)$ is strictly decreasing on I. Hence, from (18) we get that $h_* = h^*$ and, consequently, h is constant on Σ^n. Therefore, $\psi(\Sigma)$ must be contained in a slice $\{\tau\} \times M^{n+p}$. $\square$

From Theorem 2 we obtain:

Corollary 3

Let $I \times_f M_\varphi^{n+1}$ *be a weighted warped product such that* $(\log f)'' \geq 0$*, with the equality* $(\log f)'' = 0$ *holding only at isolated points of* I*, and which obeys the convergence condition* (17)*. Suppose in addition that* f' *does not vanish on* I *and* $\operatorname{Hess}\varphi$ *is bounded from below. There do not exist complete* φ*-minimal submanifolds* $\psi : \Sigma^n \to I \times_f M_\varphi^{n+1}$ *lying in a slab of* $I \times_f M_\varphi^{n+1}$ *and with bounded second fundamental form.*

The next key lemma is just an extension of a Liouville-type result due to Yau in [5], and its proof can be found in [4].

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The only φ-harmonic bounded functions defined on an n-dimensional complete weighted Riemannian manifold Σ^n_φ, whose Bakry-Émery-Ricci tensor is nonnegative, are the constant ones.

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Theorem 3

Let $I \times_f M^{n+p}_\varphi$ be a weighted warped product such that $(\log f)'' \geq 0$ and let $\psi : \Sigma^n \to I \times_f M^{n+p}_\varphi$ be a complete submanifold which lies in a slab of $I \times_f M^{n+p}_\varphi$, having nonnegative Bakry-Émery-Ricci tensor and such that the support function $\langle \vec{H}_\varphi, \partial_t \rangle$ is constant. Then, $\psi(\Sigma)$ is contained in a slice $\{\tau\} \times M^{n+p}$, for some $\tau \in I$. Moreover, when $p = 1$, $\phi := \pi_M \circ \psi : \Sigma^n \to M^{n+1}$ is a hypersurface with φ-mean curvature $H_{\phi,\varphi}$ satisfying (13).

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Corollary 4

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Taking into account Corollary 4, we can use Theorem 4 of [6] to obtain a new Bernstein-type result. In what follows a $(p+1)$-*graph* in $\mathbb{R}^{p+1} \times \mathbb{G}^n$ defined over $\mathbb{G}^n$ is a graph $u : \mathbb{G}^n \to \mathbb{R}^{p+1}$, with $(u(x), x) \in \mathbb{R}^{p+1} \times \mathbb{G}^n$.

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The only complete φ-minimal bounded $(p+1)$-graphs in $\mathbb{R}^{p+1} \times \mathbb{G}^n$ defined over $\mathbb{G}^n$, having nonnegative Bakry-Émery-Ricci tensor, are the n-dimensional hyperplanes $\{q\} \times \mathbb{G}^n$ with $q \in \mathbb{R}^{p+1}$.

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References

[1] L.J. Alías, V. Cánovas and A.G. Colares, *Marginally trapped submanifolds in generalized Robertson-Walker Spacetimes*, Gen. Relativ. Gravit. **49** (2017), 1–23.

References

[1] L.J. Alías, V. Cánovas and A.G. Colares, *Marginally trapped submanifolds in generalized Robertson-Walker Spacetimes*, Gen. Relativ. Gravit. **49** (2017), 1–23.

[2] L.J. Alías, T. Kurose and G. Solanes, *Hadamard-type theorems for hypersurfaces in hyperbolic spaces*, Diff. Geom. Appl. **24** (2006), 492–502.

[1] L.J. Alías, V. Cánovas and A.G. Colares, *Marginally trapped submanifolds in generalized Robertson-Walker Spacetimes*, Gen. Relativ. Gravit. **49** (2017), 1–23.

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[3] J.G. Araújo, H.F. de Lima and M.A.L. Velásquez, *Submanifolds immersed in a warped product: rigidity and nonexistence*, Proceedings of the American Mathematical Society, **147** (2019) , 811–821.

References

[1] L.J. Alías, V. Cánovas and A.G. Colares, *Marginally trapped submanifolds in generalized Robertson-Walker Spacetimes*, Gen. Relativ. Gravit. **49** (2017), 1–23.

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[4] K. Brighton, *A Liouville-type Theorem for Smooth Metric Measure Spaces*, J. Geom. Anal. **23** (2013), 562–570.

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[5] M. Gromov, *Isoperimetry of waists and concentration of maps*, Geom. Funct. Anal. **13** (2003), 178–215.

[1] L.J. Alías, V. Cánovas and A.G. Colares, *Marginally trapped submanifolds in generalized Robertson-Walker Spacetimes*, Gen. Relativ. Gravit. **49** (2017), 1–23.

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[6] D.T. Hieu and T.L. Nam, *Bernstein type theorem for entire weighted minimal graphs in* $\mathbb{G}^n \times \mathbb{R}$, J. Geom. Phys. **81** (2014), 87–91.

[1] L.J. Alías, V. Cánovas and A.G. Colares, *Marginally trapped submanifolds in generalized Robertson-Walker Spacetimes*, Gen. Relativ. Gravit. **49** (2017), 1–23.

[2] L.J. Alías, T. Kurose and G. Solanes, *Hadamard-type theorems for hypersurfaces in hyperbolic spaces*, Diff. Geom. Appl. **24** (2006), 492–502.

[3] J.G. Araújo, H.F. de Lima and M.A.L. Velásquez, *Submanifolds immersed in a warped product: rigidity and nonexistence*, Proceedings of the American Mathematical Society, **147** (2019) , 811–821.

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[8] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205–214.

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[9] S. Pigola, M. Rigoli and A.G. Setti, *Maximum principles on Riemannian manifolds and applications*, Mem. Amererican Math. Soc. **174** (2005), Number 822.

References

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[10] M. Rimoldi, *Rigidity results for Lichnerowicz Bakry-Émery Ricci tensor*, Ph.D. thesis, Università degli Studi di Milano, 2011.

[11] S.T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

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wfgomes@mat.ufcg.edu.br
wfgomes.uepb@gmail.com

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wfgomes@mat.ufcg.edu.br
wfgomes.uepb@gmail.com

Unidade Acadêmica de Matemática

UNIVERSIDADE FEDERAL DE CAMPINA GRANDE