Projected Ricci flow and applications to flag manifolds

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Goal: Provide a complete description of the global behavior of the homogeneous Ricci flow on flag manifolds with three isotropy summands.

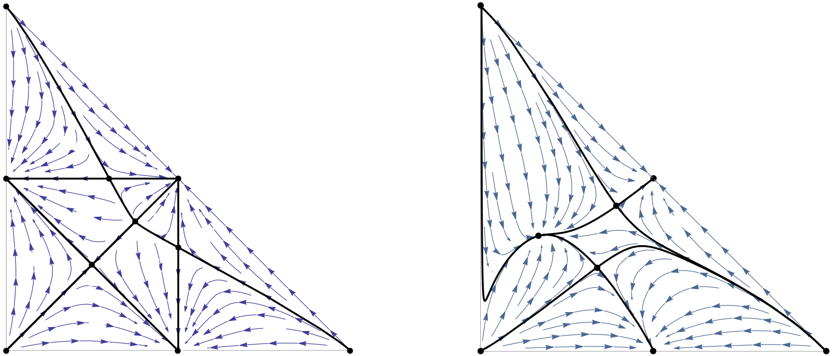


Figure: Projected Ricci flow of Type II (left) and Type I (right).

Generalized flag manifolds

Let a compact connected Lie group G have Lie algebra $\mathfrak{g}$ and a maximal torus T with Lie algebra $\mathfrak{t}$. We have that $\mathfrak{g}$ is the compact real form of the complex reductive Lie algebra $\mathfrak{g}_{\mathbb{C}}$. The adjoint representation of the Cartan subalgebra $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ splits as the root space decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha$ with root space

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_{\mathbb{C}} : \operatorname{ad}(H)X = \alpha(H)X, \forall H \in \mathfrak{h}\},$$

where $\Pi \subset \mathfrak{h}^*$ is the root system.

Consider

$$\mathfrak{m}_\alpha = \mathfrak{g} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$$

and let Π^+ be a choice of positive roots, then $\mathfrak{g}$ splits as

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Pi^+} \mathfrak{m}_\alpha .$$

Denote by Σ the subset of simple roots corresponding to Π^+.

A flag manifold of G is a homogeneous space G/K where K is the centralizer of a torus. We have that K is connected and w.l.o.g. we may assume that $T \subset K$. Recall that T is the centralizer of $\mathfrak{t}$. More generally, one can take $K = G_\Theta$ being the centralizer of

$$\mathfrak{t}_\Theta = \{H \in \mathfrak{t} : \ \alpha(H) = 0, \, \alpha \in \Theta\}$$

and Θ is a subset of the simple roots Σ which, in rough terms, furnishes the block structure of the isotropy G_Θ (recall the painting Dynking diagrams classification of flag manifolds)

The Lie algebra $\mathfrak{k} = \mathfrak{g}_\Theta$ splits as

$$\mathfrak{k} = \mathfrak{t} \oplus \sum_{\alpha \in \langle \Theta \rangle^+} \mathfrak{m}_\alpha,$$

where $\langle \Theta \rangle^+$ is the set of positive roots given by sums of roots in Θ. We denote the generalized flag manifold by

$$\mathbb{F}_\Theta = G/G_\Theta \tag{1}$$

with basepoint $b = G_\Theta$.

A G_Θ-invariant isotropy complement of $\mathbb{F}_\Theta$ is given by

$$\mathfrak{m} = \sum_{\alpha \in \Pi^+ - \langle \Theta \rangle^+} \mathfrak{m}_\alpha,$$

so that $\mathbb{F}_\Theta$, with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, is reductive and the isotropy representation of $\mathbb{F}_\Theta$ is equivalent to the adjoint representation of G_Θ in $\mathfrak{m}$. This representation is completely reducible and can be uniquely decomposed as the sum of non-equivalent irreducible representations

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_n,$$

where each $\mathfrak{m}_k$ is an appropriate sum of $\mathfrak{m}_\alpha$'s.

With this notation on mind, let us discuss about invariant metrics and invariant tensors.

Recall: there is 1-1 correspondence between G-invariant tensors on G/G_Θ and $\mathrm{Ad}(G_\Theta)$-invariant tensors on $\mathfrak{m}$.

Description of invariant metrics: an invariant metric on the flag G/G_Θ is described by a n-uple of positive numbers $g = (\lambda_1, \ldots, \lambda_n)$, being n the number of isotropy components.

More precisely,

$$g_b = x_1 B_1 + \ldots + x_n B_n \quad (2)$$

where $x_i > 0$ and B_i is the restriction of the (negative of the) Cartan-Killing form of $\mathfrak{g}$ to $\mathfrak{m}_i$, and $b = eG_\Theta$ is the trivial coset.

We also have

$$\mathrm{Ric}(g_b) = y_1 B_1 + \ldots + y_n B_n \tag{3}$$

where y_i is a function of $x_1, \ldots, x_n$.

Remark: General formula for the components of the Ricci tensor due to Wang-Ziller.

Therefore, the *homogeneous Ricci flow* becomes the autonomous system of ordinary differential equations

$$\frac{dx_k}{dt} = -2y_k, \qquad k = 1, \ldots, n. \tag{4}$$

Next, we write the Ricci flow equation in terms of the Ricci operator $r(g)_b$. Since $r(g)_b$ is invariant under the isotropy representation, $r(g)_b|_{\mathfrak{m}_k}$ is a multiple r_k of the identity. From (2) and (3), we get

$$y_k = x_k r_k$$

and equation (4) becomes

$$\frac{dx_k}{dt} = -2x_k r_k \tag{5}$$

We denote by $R(x_1, \dots, x_n)$ the vector field on the right hand side of (5), with phase space $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R} : \, x_i > 0\}$. Moreover, $x \in \mathbb{R}^n_+$ corresponds to an Einstein if and only if $R(x) = \lambda x$, for some $\lambda > 0$.

Projected Ricci Flow

One technique to study the homogeneous Ricci flow in flag manifolds (or other reductive homogeneous space) is the so called Poincaré compactiication.

This technique was used in a joint work with R. Miranda (2009) with further developments by Anastassiou, Chrysikos, do Prado, Statha, ...

Roughly speaking: under certain assumptions one can analyze the behavior ”at infinity of a dynamical system on $\mathbb{R}^{n+1}$ via an induced dynamical system on the compactification D^{n+1} (disc), including the boundary S^n.

Now we are going normalizing the flow to a simplex and time reparametrizing it to get polynomial equations, obtaining what we call the *projected Ricci flow*.

Let us consider $(x_1, \ldots, x_n) \in \mathbb{R}^n_+$. Denote

$$W(x) = \overline{x} = x_1 + \cdots + x_n.$$

The level set $W(x) = 1$ in $\mathbb{R}^n_+$ is the open canonical n-dimensional simplex $\mathcal{T}$.

Theorem 1

The solutions of the Ricci flow

$$\frac{dx}{dt} = R(x)$$

can be rescaled in space and reparametrized in time to solutions of the projected flow

$$\frac{dx}{dt} = R(x) - \overline{R(x)}x, \qquad \overline{x} = 1 \tag{6}$$

and vice-versa, where x is an equilibrium of the previous equation if and only x is Einstein with $\overline{x} = 1$. ($\overline{R(x)}$ denote the sum of the components of $R(x)$).

To study the limiting behavior of (6) on $\mathcal{T}$, it is convenient to multiply it by an appropriate positive function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ in order to get a homogeneous polynomial vector field $X(x)$ defined in the closure of $\mathcal{T}$ and tangent to the boundary of $\mathcal{T}$, given by

$$\begin{aligned} X(x) &= f(x)\left(R(x) - \overline{R(x)}\,x\right) \\ &= (fR)(x) - \overline{(fR)(x)}\,x \end{aligned} \tag{7}$$

since $W(x) = \overline{x}$ is linear.

Therefore, to get a polynomial vector field X, it suffices to choose f such that $(fR)(x) = f(x)R(x)$ is a polynomial vector field.

In order for X to be tangent to the boundary of $\mathcal{T}$, it is sufficient that the i-th coordinate of $(fR)(x)$ vanishes whenever the i-th coordinate does or, equivalently, that each coordinate hyperplane $\Pi_i = \{x : \ x_i = 0\}$ is invariant by the flow of fR.

Given a subset of indexes $I \subseteq \{1, \ldots, n\}$, consider the subspace $\Pi_I = \cap_{i \in I} \Pi_i$ and let $\mathcal{T}_I = \mathrm{cl}(\mathcal{T}) \cap \Pi_I$ be the I-th face of the simplex $\mathcal{T}$. Note that $\mathcal{T}_\varnothing = \mathrm{cl}(\mathcal{T})$.

Proposition 1

If fR is tangent to each hyperplane Π_i, then each face $\mathcal{T}_I$ of $\mathcal{T}$ is invariant by the flow of X. In particular, $\mathrm{cl}(\mathcal{T})$ is invariant and its vertices are fixed points.

One more modification...

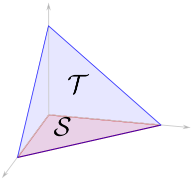


Figure: Simplexes $\mathcal{T}$ and $\mathcal{S}$ in the case of metrics with 3 parameters.

We will analyze the dynamics of the projection of X to the simplex

$$\mathcal{S} = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}_+^{n-1} : \ x_1 + \cdots + x_{n-1} \leq 1\}$$

associated to the conjugated vector field

$$Y = P \circ X \circ P^{-1}$$

where $P : \mathcal{T} \to \mathcal{S}$ is given by the projection
$P(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1})$ with inverse
$P^{-1}(x_1, \ldots, x_{n-1}) = P(x_1, \ldots, x_{n-1}, 1 - x_1 - \cdots - x_{n-1})$.

Proposition 2

If the vector field fR is polynomial of degree d, then the vector fields X given by equation (7) and $Y = P \circ X \circ P^{-1}$ are polynomial of degree $d + 1$ and the associated flows are conjugated. Moreover, $x \in \mathcal{T}$ is Einstein if and only if $Y(Px) = 0$.

The flow of Y in $\mathcal{S}$ is the so called projected Ricci flow.

Flag manifolds with three isotropy components.

There exist two classes of flag manifolds with three isotropy summands, of *Type II* and of *Type I*, depending on the Dynkin mark of the roots in $\Pi^+ \setminus \Theta^+$.

Recall that the Dynkin mark of a simple root $\alpha \in \Sigma$ is the coefficient $\mathrm{mrk}(\alpha)$ of α, in the expression of the highest root as a combination of simple roots.

The generalized flag manifold G/G_Θ has three isotropy summands if, and only if, the set $\Theta \subset \Sigma$ is given by

Type	
I	$\Sigma \setminus \Theta = \{\alpha : \mathrm{mrk}(\alpha) = 3\}$
II	$\Sigma \setminus \Theta = \{\alpha, \beta : \ \mathrm{mrk}(\alpha) = \mathrm{mrk}(\beta) = 1\}$

Classification of Einstein metrics on this family of homogeneous space due to Kimura.

Table: Complex flag manifolds with three summands of Type I

$E_8/E_6 \times SU(2) \times U(1)$
$E_8/SU(8) \times U(1)$
$E_7/SU(5) \times SU(3) \times U(1)$
$E_7/SU(6) \times SU(2) \times U(1)$
$E_6/SU(3) \times SU(3) \times SU(2) \times U(1)$
$F_4/SU(3) \times SU(2) \times U(1)$
$G_2/U(2)$

Each flag manifold of type I admits exactly three invariant Einstein metrics (up to scale); exactly one of them is Einstein-Kähler.

Table: Complex flag manifolds with three summands of Type II

$SU(m+n+p)/S(U(m) \times U(n) \times U(p))$
$SO(2\ell)/U(1) \times U(\ell-1)$, $\ell \geq 4$
$E_6/SO(8) \times U(1) \times U(1)$

Each flag manifold of type II admits exactly four invariant Einstein metrics (up to scale); exactly three of them is Einstein-Kähler.

Flag manifold of Type II

Let us describe the flag manifold $SU(m+n+p)/S(U(m)\times U(n)\times U(p))$. The analysis for the other flags is done in a similar way.

Let us denote an invariant metric by $g=(x,y,z)$.

The components of the Ricci operator of the invariant metric g are given by

$$
\begin{aligned}
r_x &= \frac{1}{2x}+\frac{mnp}{4mn(m+n+p)}\left(\frac{x}{yz}-\frac{z}{xy}-\frac{y}{xz}\right)\\
r_y &= \frac{1}{2y}+\frac{mnp}{4mp(m+n+p)}\left(\frac{y}{xz}-\frac{x}{yz}-\frac{z}{xy}\right)\\
r_z &= \frac{1}{2z}+\frac{mnp}{4np(m+n+p)}\left(\frac{z}{xy}-\frac{x}{yz}-\frac{y}{xz}\right)
\end{aligned}
$$

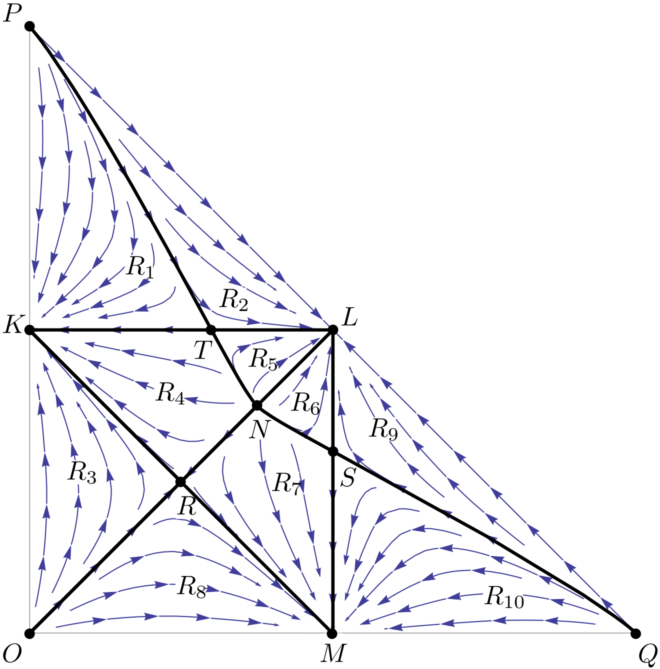
and the corresponding Ricci flow equation

$$x'=-2xr_x \qquad y'=-2yr_y \qquad z'=-2zr_z$$

The *projected Ricci flow* is given by

$$\begin{cases} u(x,y) &= -x(2x-1)(m(4y-1)(x+y-1)+ny(4x+4y-3)+p(x(4y-1)+(1-2y)^2)) \\ v(x,y) &= -y(2y-1)(m(4x-1)(x+y-1)+n(y(4x-1)+(1-2x)^2)+px(4x+4y-3)) \end{cases} \tag{8}$$

Singularity	Type of metric	λ_1	λ_2	Singularity
$O = (0,0)$	degenerate	$m+p$	$m+n$	repeller
$P = (0,1)$	degenerate	$n+p$	$m+n$	repeller
$Q = (1,0)$	degenerate	$n+p$	$m+p$	repeller
$K = (0, \frac{1}{2})$	degenerate	$-\frac{1}{2}(m+n)$	$-\frac{1}{2}(m+n)$	attractor
$L = (\frac{1}{2}, \frac{1}{2})$	degenerate	$-\frac{1}{2}(n+p)$	$-\frac{1}{2}(n+p)$	attractor
$M = (\frac{1}{2}, 0)$	degenerate	$-\frac{1}{2}(m+p)$	$-\frac{1}{2}(m+p)$	attractor
$N = \left(\frac{m+n}{2(m+n+p)}, \frac{m+p}{2(m+n+p)}\right)$	Einstein non-Kähler	$\lambda_1(N)$	$\lambda_2(N)$	repeller
$R = \left(\frac{m+n}{2(2m+n+p)}, \frac{m+p}{2(2m+n+p)}\right)$	Kähler-Einstein	$-\frac{m(m+n)(m+p)}{(2m+n+p)^2}$	$\frac{(m+n)(m+p)}{2(2m+n+p)}$	saddle
$S = \left(\frac{1}{2}, \frac{m+p}{2(m+n+2p)}\right)$	Kähler-Einstein	$-\frac{p(m+p)(n+p)}{(m+n+2p)^2}$	$\frac{(m+p)(n+p)}{2(m+n+2p)}$	saddle
$T = \left(\frac{m+n}{2(m+2n+p)}, \frac{1}{2}\right)$	Kähler-Einstein	$-\frac{n(m+n)(n+p)}{(m+2n+p)^2}$	$\frac{(m+n)(n+p)}{2(m+2n+p)}$	saddle


P
K
L
T
O
M
Q
N
R
S
R_1
R_2
R_3
R_4
R_5
R_6
R_7
R_8
R_9
R_{10}

Theorem 2

Consider the flag manifold $\mathbb{F} = SU(m+n+p)/S(U(m)\times U(n)\times U(p))$. Then the limiting behavior of the projected Ricci flow is given by

1. *the Kähler Einstein metrics (R, S and T) are hyperbolic saddles,*
2. *the non-Kähler Einstein metric (N) is a repeller,*
3. *if the metric g_0 belongs to R_1, R_3 or R_4 then $\mathbb{F}_\infty = (Gr_{m+n}(\mathbb{C}^{m+n+p}), g_{\text{normal}})$,*
4. *if the metric g_0 belongs to R_2, R_5, R_6 or R_9 then $\mathbb{F}_\infty = (Gr_{m+p}(\mathbb{C}^{m+n+p}), g_{\text{normal}})$,*
5. *if the metric g_0 belongs to R_7, R_8 or R_{10} then $\mathbb{F}_\infty = (Gr_{n+p}(\mathbb{C}^{m+n+p}), g_{\text{normal}})$,*
6. *if the metric g_0 lies outside the triangle delimited by L, T, K, R, M and the flow lines connecting them, then $\mathbb{F}_{-\infty} =$ point,*

where $\mathbb{F}_{\pm\infty} = \lim\limits_{t\to\pm\infty}(\mathbb{F}, g_t)$, g_t is the projected Ricci flow with initial condition g_0 and the convergence is in Gromov-Hausdorff sense.

Theorem 3
Consider the flag manifold $\mathbb{F} = SO(2\ell)/U(1) \times U(\ell-1)$, $\ell \geq 4$. *Then the limiting behavior of the projected Ricci flow is given by*

1. *The Einstein-Kähler metrics (R, S and T) are hyperbolic saddles,*
2. *The Einstein non-Kähler metric (N) is a repeller,*
3. *if the metric g_0 belongs to R_1, R_3 or R_4 then* $\mathbb{F}_\infty = (SO(2\ell)/U(\ell), g_{\text{normal}})$,
4. *if the metric g_0 belongs to R_2, R_5, R_6 or R_9 then* $\mathbb{F}_\infty = (SO(2\ell)/SO(2\ell-2) \times SO(2), g_{\text{normal}})$,
5. *if the metric g_0 belongs to R_7, R_8 or R_{10} then* $\mathbb{F}_\infty = (SO(2\ell)/U(\ell), g_{\text{normal}})$,
6. *if the metric g_0 lies outside the triangle delimited by L, T, K, R, M and the flow lines connecting them, then $\mathbb{F}_{-\infty}$ = point,*

where $\mathbb{F}_{\pm\infty} = \lim_{t\to\pm\infty} (\mathbb{F}, g_t)$, *$g_t$ is the projected Ricci flow with initial condition g_0 and the convergence is in Gromov-Hausdorff sense.*

Theorem 4

Consider the flag manifold $\mathbb{F} = E_6/(SO(8) \times U(1) \times U(1))$. *Then the limiting behavior of the projected Ricci flow is given by*

1. *The Einstein Kähler metrics* (R, S *and* T) *are hyperbolic saddle points,*
2. *The Einstein non-Kähler metric* (N) *is a repeller,*
3. *if the metric* g_0 *belongs to* R_1, R_3 *or* R_4 *then* $\mathbb{F}_\infty = (E_6/SO(10) \times U(1), g_{\text{normal}})$,
4. *if the metric* g_0 *belongs to* R_2, R_5, R_6 *or* R_9 *then* $\mathbb{F}_\infty = (E_6/SO(10) \times U(1), g_{\text{normal}})$,
5. *if the metric* g_0 *belongs to* R_7, R_8 *or* R_{10} *then* $\mathbb{F}_\infty = (E_6/SO(10) \times U(1), g_{\text{normal}})$,
6. *if the metric* g_0 *lies outside the triangle delimited by* L, T, K, R, M *and the flow lines connecting them, then* $\mathbb{F}_{-\infty} =$ *point,*

where $\mathbb{F}_{\pm\infty} = \lim_{t\to\pm\infty} (\mathbb{F}, g_t)$, g_t *is the projected Ricci flow with initial condition* g_0 *and the convergence is in Gromov-Hausdorff sense.*

Type I flag manifolds

Consider G/K be a flag manifold of type I and consider the decomposition of $\mathfrak{m}$ into irreducibles components

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

Denote by $d_i = \dim \mathfrak{m}_i$.

Table: Type I flag manifolds

Flag manifold	d_1	d_2	d_3
$E_8/E_6 \times SU(2) \times U(1)$	108	54	4
$E_8/SU(8) \times U(1)$	112	56	16
$E_7/SU(5) \times SU(3) \times U(1)$	60	30	10
$E_7/SU(6) \times SU(2) \times U(1)$	60	30	4
$E_6/SU(3) \times SU(3) \times SU(2) \times U(1)$	36	18	4
$F_4/SU(3) \times SU(2) \times U(1)$	24	12	4
$G_2/U(2)$	4	2	4

Given an invariant metric $g = (x, y, z)$ the components of the Ricci operator are given by:

$$\begin{aligned} r_x &= \frac{y(-d_1 d_2 - 2d_1 d_3 + d_2 d_3)}{2x^2 d_1 (d_1 + 4d_2 + 9d_3)} + \frac{d_3 (d_1 + d_2)}{2d_1 (d_1 + 4d_2 + 9d_3)} \left(\frac{x}{yz} - \frac{z}{xy} - \frac{y}{xz} \right) + \frac{1}{2x} \\ r_y &= -\frac{(-d_1 d_2 - 2d_1 d_3 + d_2 d_3)}{4d_2 (d_1 + 4d_2 + 9d_3)} \left(\frac{y}{x^2} - \frac{2}{y} \right) + \frac{d_3 (d_1 + d_2)}{2d_2 (d_1 + 4d_2 + 9d_3)} \left(-\frac{x}{yz} - \frac{z}{xy} + \frac{y}{xz} \right) + \frac{1}{2y} \\ r_z &= \frac{(d_1 + d_2)}{2(d_1 + 4d_2 + 9d_3)} \left(-\frac{x}{yz} + \frac{z}{xy} - \frac{y}{xz} \right) + \frac{1}{2z} \end{aligned}$$

together with the corresponding Ricci flow equation

$$x' = -2x r_x \qquad y' = -2y r_y \qquad z' = -2z r_z$$

The projected Ricci flow equations are given by:

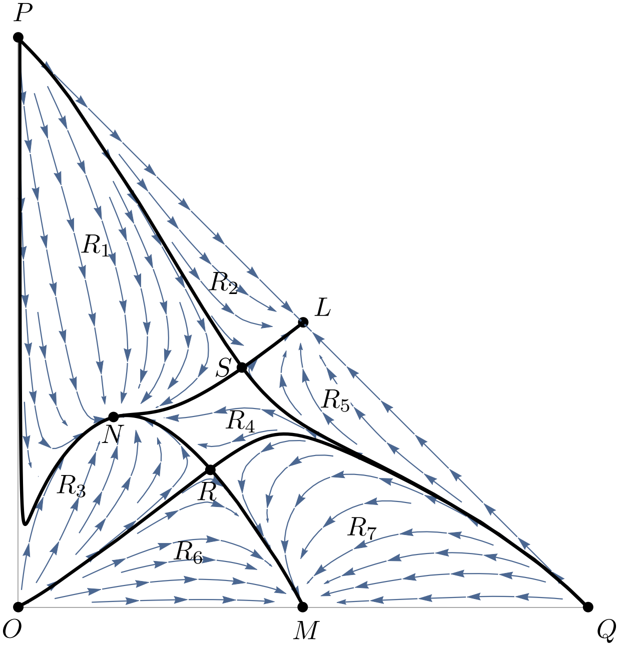
$$\begin{cases} u(x,y) &= x(-4d_2^2 d_3(2x^3(-1+y)-(-1+y)y^2+x^2(3-4y+3y^2)+x(-1+2y-4y^2+y^3)) \\ & -2d_1^2(2d_3(-1+y)y(x(-1+y)+y^2)+d_2((-1+y)y^3+x^3(-4+8y) \\ & +2x^2(3-9y+4y^2)+x(-2+8y-6y^2+y^3)))+2d_1d_2(-2d_2((-1+y)y^2 \\ & +2x^3(-1+7y)+x^2(3-22y+13y^2)-x(1-7y+4y^2+y^3)) \\ & +d_3(x^3(4-64y)+x^2(-6+86y-60y^2)+y^2(4-5y+y^2) \\ & +x(2-24y+18y^2+5y^3)))) \\ \\ v(x,y) &= y(-4d_2^2 d_3 x(2x^2(-1+y)+(-1+y)y^2+x(1-2y+3y^2)) \\ & -2d_1^2(2d_3(-1+y)^2(x(-1+y)+y^2)+d_2((-1+y)^2y^2+x^3(-4+8y) \\ & +2x^2(3-8y+4y^2)+x(-2+6y-5y^2+y^3)))+2d_1d_2(2d_2x(1+x^2(6-14y) \\ & -3y+y^2+y^3+x(-7+22y-13y^2))+d_3(x^3(28-64y)+(-1+y)^2y^2 \\ & +x^2(-26+88y-60y^2)+x(2-6y-y^2+5y^3)))) \end{cases}$$

Theorem 5

Consider the flag manifolds of Type I and its corresponding projected Ricci flow equations. We have

1. *degenerate metrics:* $O = (0,0)$, $P = (0,1)$, $Q = (1,0)$ *are repellers and* $L = (\frac{1}{2}, \frac{1}{2})$, $M = (\frac{1}{2}, 0)$ *are attractors.*
2. *Einstein-Kähler metric:* $N = (\frac{1}{6}, \frac{1}{3})$ *is an attractor.*
3. *Einstein non-Kähler metrics* R, S *are hyperbolic saddles, they depend on* d_1, d_2 *and* d_3 *and are given in the following table (in decimal approximation)*

Flag Manifold G/K	R	S
$E_8/E_6 \times SU(2) \times U(1)$	$(0.46847, 0.47077)$	$(0.28932, 0.26453)$
$E_8/SU(8) \times U(1)$	$(0.33648, 0.24145)$	$(0.39343, 0.42039)$
$E_7/SU(5) \times SU(3) \times U(1)$	$(0.33218, 0.24367)$	$(0.39938, 0.42346)$
$E_7/SU(6) \times SU(2) \times U(1)$	$(0.44544, 0.45244)$	$(0.30245, 0.25819)$
$E_6/SU(3) \times SU(3) \times SU(2) \times U(1)$	$(0.32220, 0.24866)$	$(0.41388, 0.43154)$
$F_4/SU(3) \times SU(2) \times U(1)$	$(0.34725, 0.23562)$	$(0.37927, 0.41362)$
$G_2/U(2)$	$(0.21154, 0.35427)$	$(0.46117, 0.08619)$


P
L
S
N
R
O
M
Q
R_1
R_2
R_3
R_4
R_5
R_6
R_7

Theorem 6

Let $\mathbb{F} = G/K$ be of Type I. Then the limiting behavior of the projected Ricci flow is given by

1. *if $g_0 \in R_6$ or R_7 then $\mathbb{F}_\infty$ is the corresponding symmetric space G/H listed in Table 4, equipped with the normal metric (up to scale),*
2. *if $g_0 \in R_2$ or R_5 then $\mathbb{F}_\infty$ is the corresponding Borel-de Siebenthal homogeneous space G/H listed in Table 5, equipped with the normal metric (up to scale),*
3. *if the metric g_0 lies outside the the cusp made up by L, S, N, M and the flow lines connecting them, then $\mathbb{F}_{-\infty} =$ point,*

where $\mathbb{F}_{\pm\infty} = \lim_{t \to \pm\infty} (\mathbb{F}, g_t)$, g_t is the projected Ricci flow with initial condition g_0 and the convergence is in Gromov-Hausdorff sense.

Table: 4

Flag manifold G/K	Symmetric space G/H	$\dim G/H$
$E_8/E_6 \times SU(2) \times U(1)$	$E_8/(E_7 \times SU(2))$	112
$E_8/SU(8) \times U(1)$	$E_8/Spin(16)$	128
$E_7/SU(5) \times SU(3) \times U(1)$	$E_7/SU(8)$	70
$E_7/SU(6) \times SU(2) \times U(1)$	$E_7/(SO(12) \times SU(2))$	64
$E_6/SU(3) \times SU(3) \times SU(2) \times U(1)$	$E_6/(SU(6) \times SU(2))$	40
$F_4/SU(3) \times SU(2) \times U(1)$	$F_4/Sp(3) \times SU(2)$	28
$G_2/U(2)$	$G_2/SO(4)$	8

Table: 5

Flag manifold G/K	G/H (Borel-de Siebenthal)	$\dim G/H$
$E_8/E_6 \times SU(2) \times U(1)$	$E_8/(E_6 \times SU(3))$	162
$E_8/SU(8) \times U(1)$	$E_8/SU(9)$	168
$E_7/SU(5) \times SU(3) \times U(1)$	$E_7/(SU(6) \times SU(3))$	90
$E_7/SU(6) \times SU(2) \times U(1)$	$E_7/(SU(6) \times SU(3))$	90
$E_6/SU(3) \times SU(3) \times SU(2) \times U(1)$	$E_6/(SU(3) \times SU(3) \times SU(3))$	54
$F_4/SU(3) \times SU(2) \times U(1)$	$F_4/(SU(3) \times SU(3))$	36
$G_2/U(2)$	$G_2/SU(3)$	6

Remarks:

- The analysis of homogeneous Ricci flow (including collapses) on homogeneous spaces with two isotropy components (not only flags) is due to Buzano.

- Our analysis agree with recent results of Lauret-Will and is related to the work Anastassiou-Chrysikos.

Thank you.