Riemannian foliations and groupoids

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Motivation

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$(M, \mathcal{F})$ regular foliation

$M = \bigcup_x L_x$

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$(M, \mathcal{F})$ regular foliation $\quad \Rightarrow \quad \mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ holonomy groupoid

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$G \curvearrowright M$ action

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$(M, \mathcal{F})$ regular foliation $\Rightarrow$ $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ holonomy groupoid

$G \curvearrowright M$ action $\Rightarrow$ $G \ltimes M \rightrightarrows M$ action groupoid

$(M, \mathcal{F})$ singular foliation $\Rightarrow$??

Motivation

$(M, \mathcal{F})$ regular foliation	$\Rightarrow$	$\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ holonomy groupoid
$G \curvearrowright M$ action	$\Rightarrow$	$G \ltimes M \rightrightarrows M$ action groupoid
$(M, \mathcal{F})$ singular foliation $+$Riemannian	$\Rightarrow$	??

Goals

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- What is the linear holonomy groupoid of a SRF?
- Why do we look for groupoids?

What is a SRF?

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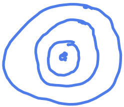
Geometric "smooth" singular foliation

$M = \bigcup L_x$, $v \in T_x L \Rightarrow \exists X \in \mathfrak{X}(U)$

com $X(x) = v$ e $X(y) \in T_y L_y$

$\mathbb{R}^2$

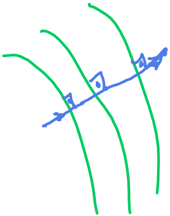


What is a SRF?

Geometric “smooth” singular foliation

Singular Riemannian foliation

$(M, \mathcal{F}, g)$



Examples

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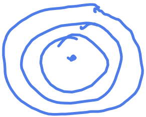
1) Isometric actions;

$G \curvearrowright (M, g)$ $\qquad$ $\mathcal{F} = \{ \mathcal{O}_x \}$

$S^1 \curvearrowright (\mathbb{R}^2, g^{can})$



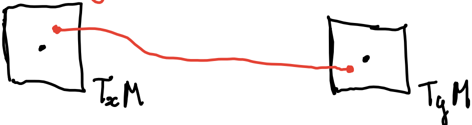
Examples

1) Isometric actions;

2) Parallel transport.

(M, g) Riemanniana $\Rightarrow$ (TM, g^{sazaki})

$v \in T_xM$, $L_v = \{P_\alpha^t(v) : \alpha \in C^\infty([0,1], M), \alpha(0) = x\}$ $\Rightarrow$ $\mathcal{F} = \{L_v\}$



Semi-local model

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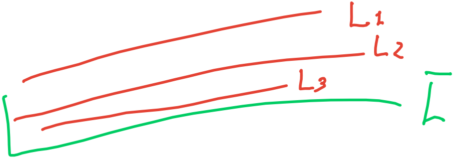
- $\mathcal{F}^B = \mathcal{F}|_B$ regular foliation by dense leaves;



Semi-local model

- $(M, \mathcal{F}, \eta)$ a SRF, and $B = \overline{L}$ closure of a leaf of $\mathcal{F}$;

- $\mathcal{F}^B = \mathcal{F}|_B$ regular foliation by dense leaves;

- $E = NB \to B$ normal bundle;

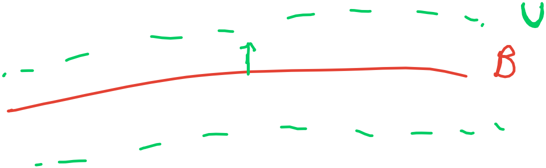
$B \subset M \qquad NB = TM|_B / TB \cong TB^{\perp}$

Semi-local model

$B = \overline{L}$

- $E \cong U \subset M$ tubular neighbourhood, and $\mathcal{F}$ induces a SRF $\mathcal{F}^E$ on E;

$(E, \varphi^*\mathcal{F}) \xrightarrow{\varphi} (U, \mathcal{F})$



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- $E \cong U \subset M$ tubular neighbourhood, and $\mathcal{F}$ induces a SRF $\mathcal{F}^E$ on E;

- Geometric description of $\mathcal{F}^E$;

$(E, g^{sasaki}, \mathcal{F}^E)$

Semi-local model

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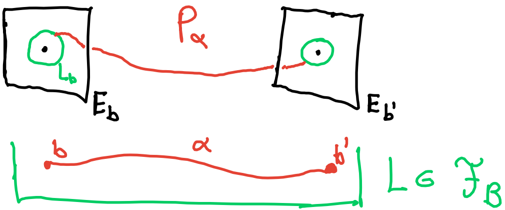
$(E_b, \mathcal{F}_b)$ $\mathcal{F}_b \simeq \mathcal{F}^E|_{E_b}$, existe $\nabla^\tau : \mathfrak{X}(\mathcal{F}_B) \times \Gamma(E) \to \Gamma(E)$ conexão parcial compatível com a métrica e preservando as folheações $\mathcal{F}_b$



Linearized foliation

- Linearize vector fields X in $\mathfrak{X}(\mathcal{F}^{E})$ along B;

$h_\lambda : E \to E$

$h_\lambda(v) = \lambda v$

$$X^l(v) = \lim_{\lambda \to 0} (h_\lambda^{-1})_* X(h_\lambda(v))$$

Ex: $(\mathbb{R}^n, \mathcal{F})$ $\quad L = \{0\}$

$$X^l(v) = \lim_{\lambda \to 0} (h_\lambda^{-1})_* X(h_\lambda(v)) = \lim_{\lambda \to 0} \frac{1}{\lambda} X(\lambda v) = (\nabla_v X)_0$$

$X(v) = Av \Rightarrow X^l = X$

Linearized foliation

- Linearize vector fields X in $\mathfrak{X}(\mathcal{F})$ along B;

- Linearized fields give rise a new SRF $\mathcal{F}^{E^\ell}$;

$$\mathfrak{X}(\mathcal{F}^E) \xrightarrow{\ \ell\ } \mathfrak{X}(M) \supset \mathfrak{X}(\mathcal{F}^{E^\ell})$$

$$\mathcal{F}^{E^\ell} \subset \mathcal{F}^E$$

$$\{X_i^\ell\} \qquad \varphi_{t_1}^{X_1^\ell}\circ \cdots \circ \varphi_{t_n}^{X_n^\ell}(x)$$

Linearized foliation

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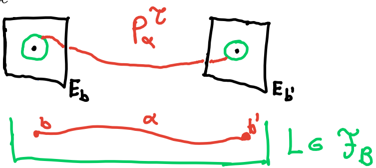
- Geometric description of $\mathcal{F}^{E^\ell}$



A few facts about Lie groupoids

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A **Lie groupoid** $G \rightrightarrows M$ is a pair of manifolds G, M together with:

- submersions $s, t : G \to M$;

- smooth partial multiplication $m : G \,{}_s\!\times_t G \xrightarrow{m} G$ admiting smooth units $u : M \to G$ and smooth inverses $i : G \to G$.

Examples

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1) Actions; $G \curvearrowright M$

$$G \times M \underset{t}{\overset{s}{\rightrightarrows}} M$$

$$s(g, x) = x$$

$$t(g, x) = gx$$

Examples

1) Actions;

2) Foliations(regular);

não é subvariedade

$(M, \mathcal{F})$ $\quad R(\mathcal{F}) \subset M \times M$ rel. eq.

$$\begin{array}{ccc} R(\mathcal{F}) & \subset & M \times M \\ \downarrow\downarrow & & \downarrow\downarrow \\ M & & M \end{array}$$

imersão

$$\mathrm{Hol}\,\mathcal{F} \to M \times M$$

$$\begin{array}{c} \mathrm{Hol}\,\mathcal{F} \\ \downarrow\downarrow \\ M \end{array}$$

Linear holonomy groupoid

$$\begin{array}{ccc} \mathrm{Hol}^{\tilde{\mathcal{F}}} & \rightrightarrows & (\mathcal{O}(E), \tilde{\mathcal{F}}) \\ \downarrow & & \downarrow \\ \mathrm{Hol}^{\tilde{\mathcal{F}}}/O_q & \rightrightarrows & (B, \mathcal{F}_B) \end{array}$$

$$T_\xi \tilde{L} = H^\tau_\xi \oplus \mathrm{Hol}_\xi$$

$\dim \tilde{L}_\xi$ é constante

$$\begin{array}{ccc} \mathrm{Hol}^{\tilde{\mathcal{F}}}/O_q & \curvearrowright & E \\ \downarrow\downarrow & & \\ B & & \end{array}$$

$$E_b \xrightarrow{\;g\;} E_{b'}$$

Linear holonomy groupoid

$\mathrm{Hol}\,\mathcal{F}^\ell \rightrightarrows E$ "estendendo" o transporte paralelo de ∇^γ

Theorem (Alexandrino, Inagaki, Struchiner, -)

$\mathrm{Hol}(\mathcal{F}^\ell) \rightrightarrows E$ *is a Lie groupoid with orbit foliation* $\mathcal{F}^\ell$

Closure Regular Riemannian foliations

$$
\begin{array}{ccc}
\overline{\mathrm{Hol}\,\mathcal{F}} & \rightrightarrows & (O(E), \overline{\tilde{\mathcal{F}}}) \\
\downarrow & & \downarrow \\
\overline{\mathrm{Hol}\,\mathcal{F}}/O_q = \bar{\mathcal{G}}^{\ell} & \rightrightarrows & (B, \mathcal{F}_B)
\end{array}
$$

$$
\begin{array}{ccc}
\bar{\mathcal{G}}^{\ell} & \curvearrowright & E \\
\Downarrow & \swarrow & \\
B & &
\end{array}
$$

Closure Regular Riemannian foliations

Theorem

If $\mathcal{F}$ is a regular Riemannian foliation on (M, η). Then there exists a proper Lie groupoid $\overline{\mathrm{Hol}(\mathcal{F})}$ whit orbit foliation $\overline{\mathcal{F}}$. Moreover, $\mathrm{Hol}(\mathcal{F})$ is a dense subgroupoid of $\overline{\mathrm{Hol}(\mathcal{F})}$.

Closure of the Linear holonomy

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Theorem (Alexandrino, Inagaki, Struchiner, -)

$\overline{\mathrm{Hol}(\mathcal{F}^\ell)}$ *is a Lie groupoid whith orbit foliation* $\overline{\mathcal{F}^\ell}$. *Moreover,* $\mathrm{Hol}(\mathcal{F}^\ell)$ *is a dense subgroupoid of* $\overline{\mathrm{Hol}(\mathcal{F}^\ell)}$.

Further directions

- Understand the existence or non-existence of a groupoid describing the non-linear part of the foliation.
- Consider closure of Lie groupoids preserving metrics.(with Ivan Struchiner)
- Use the groupoid to perform deformations on the metrics such as Cheeger deformations of isometric actions.(with Leonardo Cavenaghi and Llohann Sperança)
- New invariants associated to SRFs and their transverse geometry, for instance, new characteristic classes.(with Dirk Toben)

Thanks!